Math 3070 § 1.	**Failure Time Example: Hypothesis**	Name: Example
Treibergs	**test for Exponential Variables**	July 10, 2011

How to test hypotheses with exponential variables is discussed in Problem 8.85 of Devore, *Probability and Statistics for Engineering and the Sciences, 8th ed.,* Brooks Cole, 2012. Devore applies it to fake data of time to failure of identical components.

Assume that we have a random sample $X_1, X_2, \ldots, X_n \sim \text{Exp}(\lambda)$. Since $\text{E}(X) = \mu = 1/\lambda$, then an estimator for μ is the mean.

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The MLE for λ turns out to be $\hat{\lambda} = 1/\hat{\mu}$.

Note that if $X_i \sim \text{Exp}(\lambda)$, then its pdf for $x \ge 0$ is

$$f(x) = \lambda e^{-\lambda x}.$$

Hence, the pdf for $Y = 2\lambda X_i$ is

$$f_Y(x) = \frac{1}{2\lambda} f\left(\frac{x}{2\lambda}\right) = \frac{1}{2}e^{-x/2} = f(x;2)$$

which is exactly the pdf for the χ^2 distribution with two degrees of freedom. For ν degrees of freedom, the χ^2 pdf is

$$f(x,\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, & \text{if } x \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is a fact that the sum of χ^2-variables is also a χ^2 variable. To see this, suppose $Y \sim \text{ChiSq}(\nu)$ and $Z \sim \text{ChiSq}(\kappa)$ so if $W = Y + Z$ then for $w \ge 0$ the cdf

$$\begin{aligned} \text{P}(W \le w) &= \text{P}(X + Y \le w) \\ &= \iint_{x+y\le w} f(x;\lambda)\, f(y;\mu)\, dx\, dy \\ &= \frac{1}{2^{(\nu+\kappa)/2}\Gamma(\nu/2)\Gamma(\kappa/2)} \int_0^w \int_0^{w-x} x^{\nu/2-1} y^{\kappa/2-1} e^{-(x+y)/2}\, dy\, dx \end{aligned}$$

Changing variables $u = x - y$, $v = x + y$ we find

$$\text{P}(W \le w) = \frac{1}{2^{(\nu+\kappa)/2}\Gamma(\nu/2)\Gamma(\kappa/2)} \int_{v=0}^{w}\int_{u=-v}^{v} \left(\frac{u+v}{2}\right)^{\nu/2-1} \left(\frac{v-u}{2}\right)^{\kappa/2-1} e^{-v/2}\, \frac{du\, dv}{2}$$

Hence, the pdf for $w \ge 0$ is

$$\begin{aligned} f_W(w) &= \frac{d}{dw}\text{P}(W \le w) \\ &= \frac{1}{2^{(\nu+\kappa)/2+1}\Gamma(\nu/2)\Gamma(\kappa/2)} \int_{u=-w}^{w} \left(\frac{u+w}{2}\right)^{\nu/2-1} \left(\frac{w-u}{2}\right)^{\kappa/2-1} e^{-w/2}\, du \\ &= \frac{w^{(\nu+\kappa)/2-2}e^{-w/2}}{2^{(\nu+\kappa)/2+1}\Gamma(\nu/2)\Gamma(\kappa/2)} \int_{u=-w}^{w} \left(\frac{u+w}{2w}\right)^{\nu/2-1} \left(\frac{w-u}{2w}\right)^{\kappa/2-1} du \\ &= \frac{w^{(\nu+\kappa)/2-2}e^{-w/2}}{2^{(\nu+\kappa)/2+1}\Gamma(\nu/2)\Gamma(\kappa/2)} \cdot \frac{2w\Gamma(\nu/2)\Gamma(\kappa/2)}{\Gamma((\nu+\kappa)/2)} \\ &= \frac{1}{2^{(\nu+\kappa)/2}\Gamma((\nu+\kappa)/2)} w^{(\nu+\kappa)/2-1} e^{-w/2} \end{aligned}$$

where we have used the fact that the integrand is up to gamma function factors the pdf of the beta distribution with $\alpha = \nu/2$, $\beta = \kappa/2$, $A = -w$ and $B = w$. Note that the pdf of the sum is also χ^2 with $f_W(w) = f(w; \nu + \kappa)$. By induction, it follows that

$$2\lambda \sum_{i=1}^{n} X_i \sim \text{ChiSq}(2n).$$

To test the hypothesis that $\lambda < \lambda_0$ it is equivalent to test $\mu > \mu_0$ where $\mu_0 = 1/\lambda_0$. Under the null hypothesis $\mu = \mu_0$, the statistic

$$X^2 = 2\lambda_0 \sum_{i=1}^{n} X_i \sim \text{ChiSq}(2n).$$

Thus there is evidence at the $\alpha = .10$ level that $\lambda < \lambda_0$ if $X^2 > \chi^2_\alpha$, where the critical value $\text{P}(\chi^2 > \chi^2_\alpha) = \alpha$. In case $n = 10$, so for 20 degrees of freedom, $\chi^2_{.10} = 28.41$. Because the statistic works out to be $X^2 = 12.44$, we cannot reject the null hypothesis: there is no strong evidence that $\lambda < \lambda_0$. We can also compute the p-value. For the one-sided upper tailed test, the p-value is $\text{P}(\chi^2 \geq X^2) = .480$. We plot the distribution with 20 degrees of freedom, the observed X^2, the critical value $\chi^2_{.10}$, the p-value (which is the entire yellow tail area under the pdf to the right of the observed X^2, and the pink area equal to .10 of the whole tail to the right of $\chi^2_{.10}$.

We simulate $B = 10,000$ random samples of size $n = 10$ taken from Exp(1). We plot the histogram of twice the sums, and the theoretical χ^2 distribution with 20 degrees of freedom. Note that the empirical distribution follows the theoretical closely.

We also test if the data is exponential and the simulated sums follow χ^2. To do this, we make QQ-plots of the data and the simulation. If there are n observations, we plot the theoretical quantile $q_i = \text{P}(p_1)$ with the observed quantile, which is the ith point y'_i, after sorting the data $y_1, \ldots, y_n$. The percentage is computed by the function $p_i =$ `ppoints(n)[i]`, which gives the fraction corresponding to the ith of n points. Devore uses the function $p_i = (i - .5)/n$ which is the same as `ppoints(n)[i]` if $n > 10$ but `ppoints(n)[i]` $= (i - .375)/(n + .25)$ if $n \leq 10$. We plot the data vs. exponential quantiles. The agreement is pretty good for such a small sample so that exponentialness is not contradicted. We also run $B = 500$ sums of exponentials and plot them against χ^2 quantiles. The points line up reasonably well, so χ^2-ness is not contradicted.

R Session:

```
R version 2.10.1 (2009-12-14)
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'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.31 (5538) powerpc-apple-darwin8.11.1]
[Workspace restored from /Users/andrejstreibergs/.RData]
```
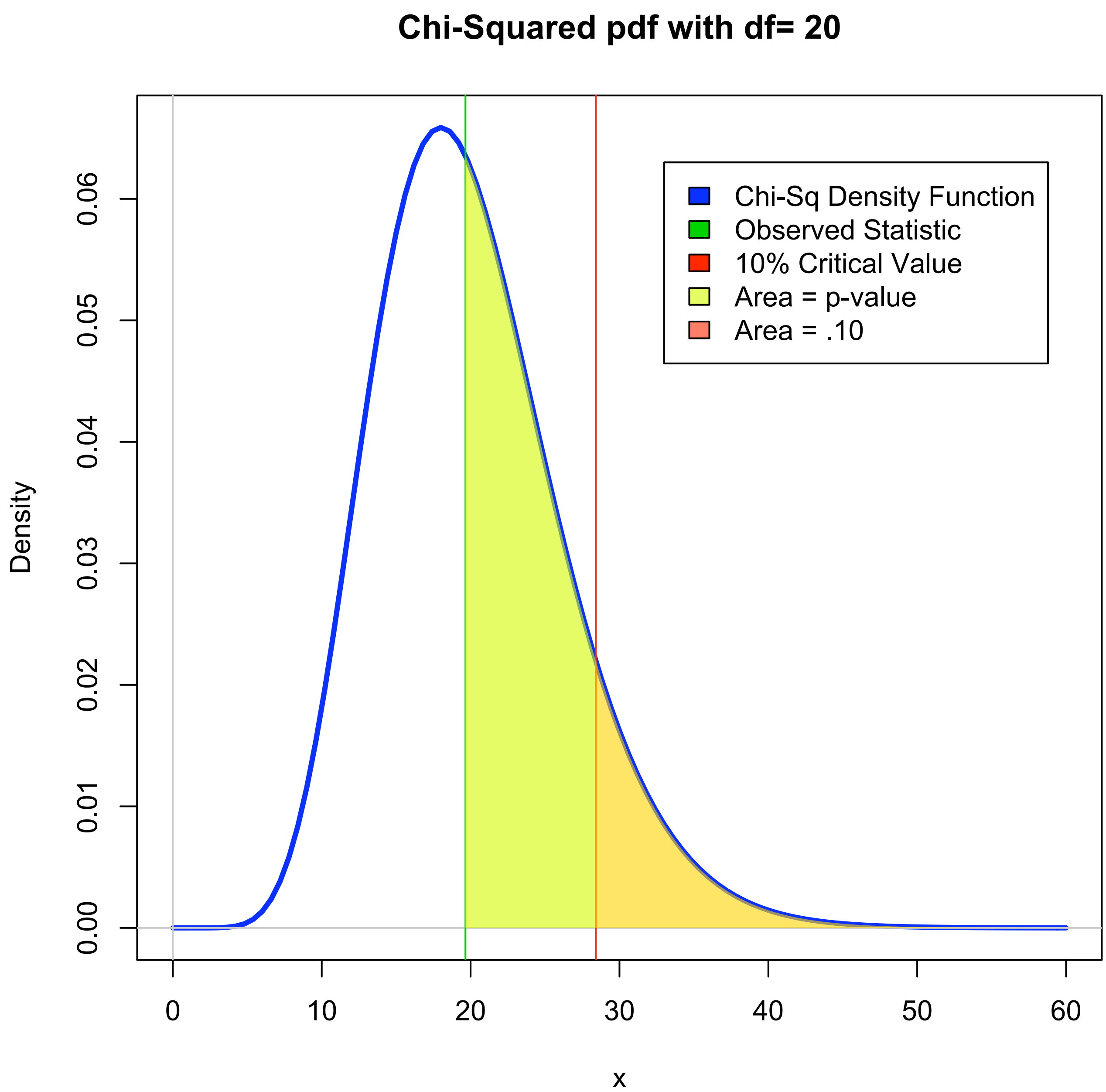
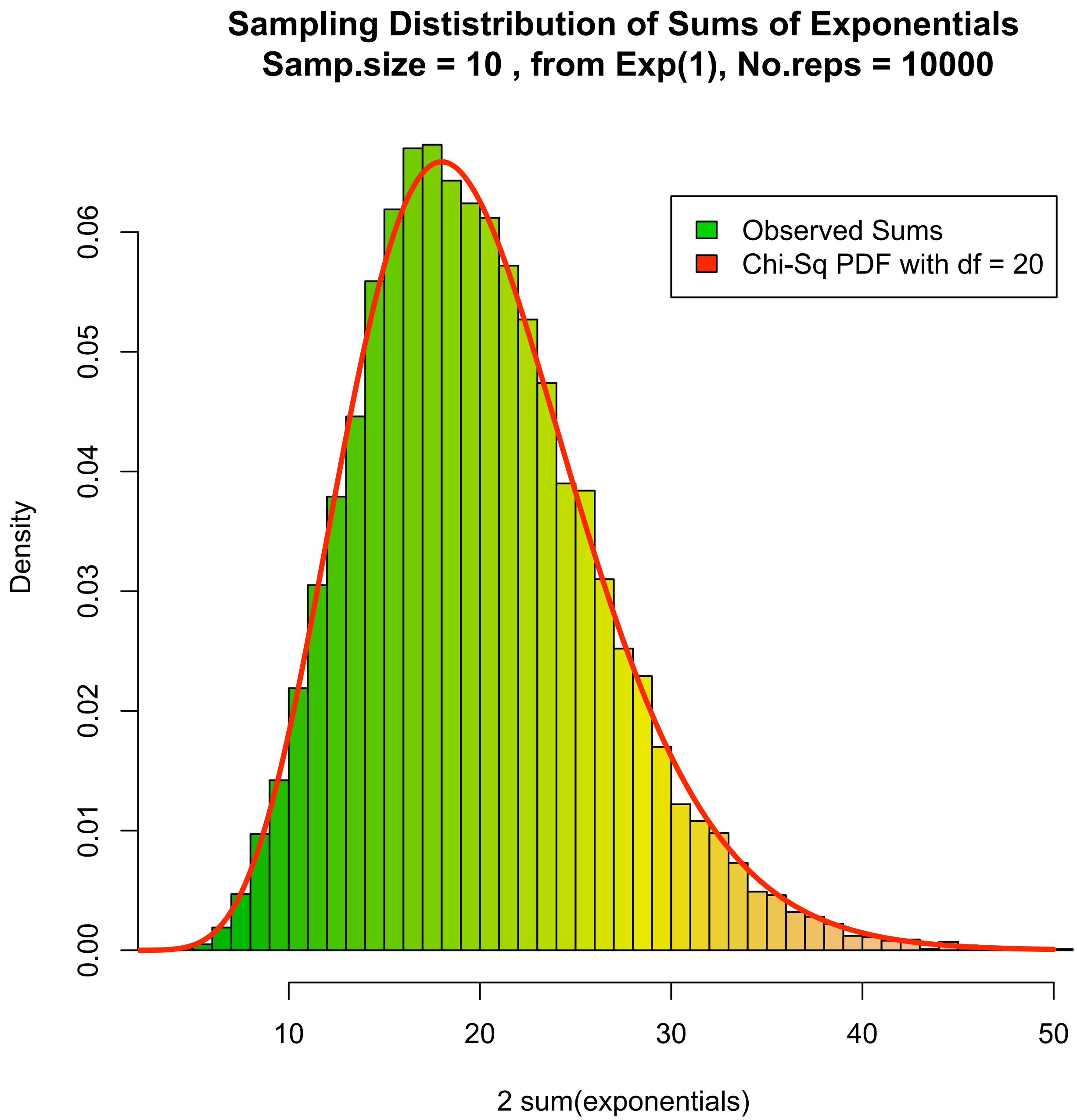
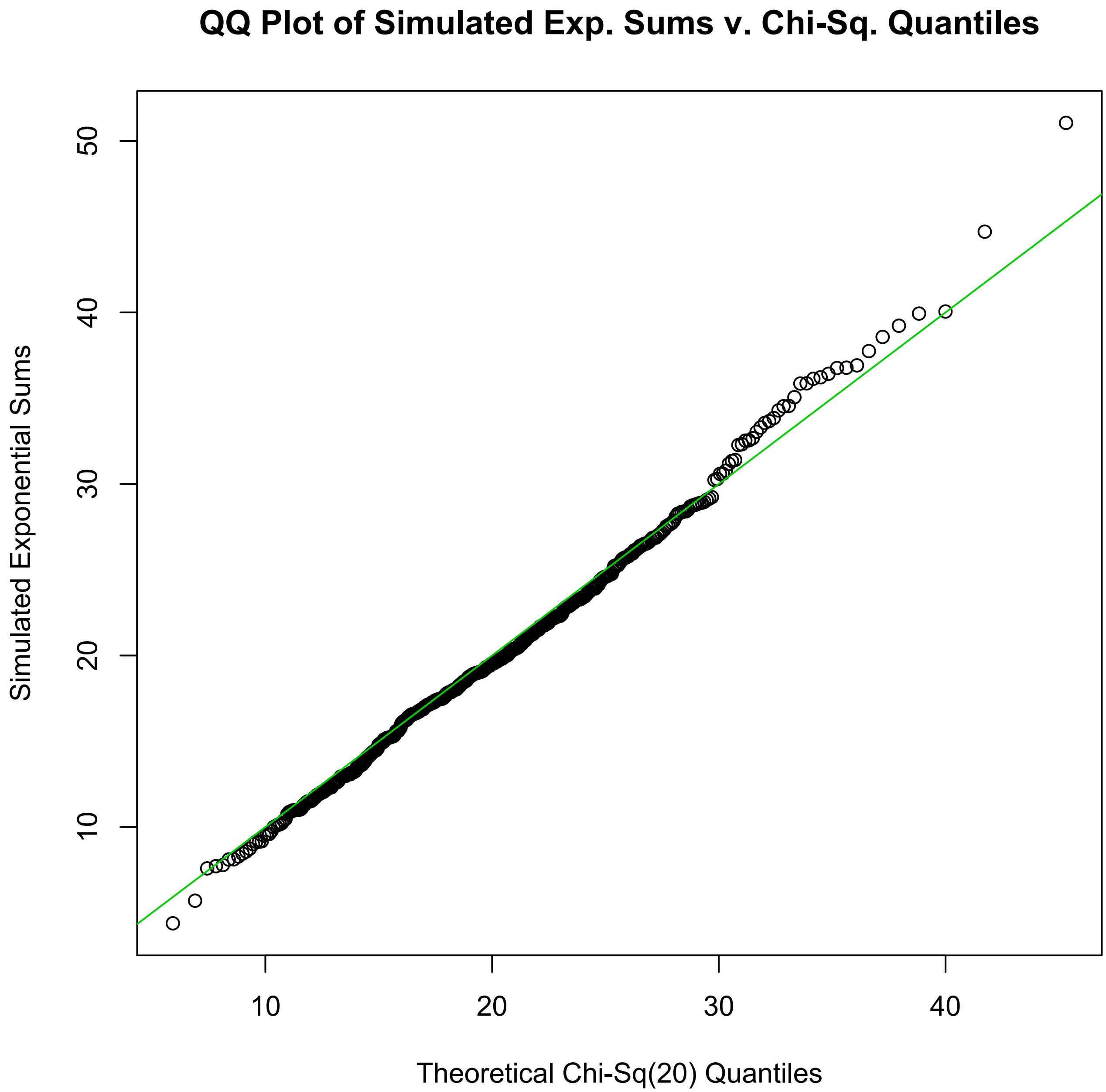

```
> ############### ENTER FAILURE TIME DATA. TEST IF LAMBDA < 75 ##################
> # Devore 8.85
> tf <- scan()
1: 95 16 11 3 42 71 225 64 87 123
11:
Read 10 items
> tf
 [1]  95  16  11   3  42  71 225  64  87 123

> # estimator for  mu
> mu <- mean(tf);mu
[1] 73.7
> lambda <- 1/mu
> mu0 <- 75
> lambda0 <- 1/mu0
> # test if mu < 75
> # equiv test if lambda > lambda0
> n <- length(tf)
> # In H0 lambda=lambda0
> # Compute the statistic
> chi2 <- 2*lambda0*sum(tf)
> chi2
[1] 19.65333

> # p - value
> pvalue <- pchisq(chi2,df=2*n,lower.tail=F); pvalue
[1] 0.4797945

> # Cannot reject niull hypothesis.
> # Find alpha = .10 critical value of statistic
> alpha <- .10
> qcrit <- qchisq(alpha,df=2*n,lower.tail=F); qcrit
[1] 28.41198

> # Larger than statistic. We cannot reject H0.
>
> ############### PLOT CHI SQ DISTRIBUTION WITH DF=20 ############################
> seq1 <- seq(chi2,qcrit,(qcrit-chi2)/117)
> seq2 <- seq(qcrit,60,(60-qcrit)/223)
> clr <- rainbow(15,alpha=.6)
> curve( dchisq(x, df = 2*n), 0, 60, lwd = 3, col = 4, ylab = "Density",
+ main = paste("Chi-Squared pdf with df=", 2*n))
> abline(h = 0, col = 8); abline(v = 0, col = 8)
> abline(v = chi2, col = 3); abline(v = qcrit, col = 2)
> polygon(c(chi2, seq1, qcrit, chi2), c(0, dchisq(seq1, df = 2*n), 0, 0),
+ col = clr[4], border = NA)
> polygon(c(qcrit, seq2, 60, qcrit), c(0, dchisq(seq2, df = 2*n), 0, 0),
+ col = clr[3], border = NA)
> legend(33, .063, legend=c("Chi-Sq Density Function", "Observed Statistic",
+ "10% Critical Value", "Area = p-value", "Area = .10"),
+ fill = c(4, 3, 2, clr[4], clr[1]), bg = "white")
> # M3074FailTime1.pdf
```


Chi-Squared pdf with df= 20
Chi-Sq Density Function
Observed Statistic
10% Critical Value
Area = p-value
Area = .10
Density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0
10
20
30
40
50
60
x

```
> ################ SAMPLING DISTRIBUTION OF SUMS OF EXPONENTIALS #################
>
> tn <- 2*n
> B <- 10000
> v <- replicate(B,2*sum(rexp(n,1)))
> hist(v, breaks = 40, freq = F, col=terrain.colors(50), xlab="2 sum(exponentials)",
+ main = paste("Sampling Dististribution of Sums of Exponentials\n Samp.size =", n,
+ ", from Exp(1), No.reps =", B))
> lines(xs, dchisq(xs, df = tn), lwd = 3, col = 2)
> legend(30, .063, legend = c("Observed Sums",paste("Chi-Sq PDF with df =", tn)),
+ fill = c(3, 2))
> # M3074FailTime2.pdf
>
> ################ EXPONENTIAL PP-PLOT TO TEST IF DATA IS EXPONENTIAL #############
>
> ppoints(n)
 [1] 0.06097561 0.15853659 0.25609756 0.35365854 0.45121951 0.54878049 0.64634146
 [8] 0.74390244 0.84146341 0.93902439

> qqplot(qexp(ppoints(n), lambda), tf, ylab="Observed Waiting Times",
+ xlab = "Theoretical Exponential Quantiles",
+ main = "QQ Plot of Waiting Times v. Exponential Quantiles")
> abline(0,1,col=2)
> # M3074FailTime3.pdf
>
> ############# CHI-SQ PP-PLOT OF SAMPLING DISTRIBUTION ###########################
>
> B <- 500
> v <- replicate(B, 2*sum(rexp(n, 1)))
> qqplot(qchisq(ppoints(B), df = tn), v, ylab="Simulated Exponential Sums",
+ xlab = "Theoretical Chi-Sq(20) Quantiles",
+ main = "QQ Plot of Simulated Exp. Sums v. Chi-Sq. Quantiles")
> abline(0,1,col=3)
> # M3074FailTime4.pdf
```


Sampling Dististribution of Sums of Exponentials
Samp.size = 10 , from Exp(1), No.reps = 10000
Observed Sums
Chi-Sq PDF with df = 20
Density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
10
20
30
40
50
2 sum(exponentials)


QQ Plot of Waiting Times v. Exponential Quantiles
Observed Waiting Times
Theoretical Exponential Quantiles
0
50
100
150
200
0
50
100
150
200


QQ Plot of Simulated Exp. Sums v. Chi-Sq. Quantiles
Simulated Exponential Sums
10
20
30
40
50
Theoretical Chi-Sq(20) Quantiles