

1. A process in the Cornish Coatings plant produces a film whose thickness is normally distributed with mean 110.0 microns and a standard deviation of 10.0 microns. The minimum acceptable thickness is 90.0 microns. What proportion of films are too thin? If the standard deviation remains 10.0 microns, to what value should the mean be set so that only 1% of the films will be too thin?

We are given that the thickness is a normal random variable $X \sim N(110.0, 10.0)$. Standardizing, the probability that the coating is too thin is

$$P(X \leq 90.0) = P\left(Z = \frac{X - \mu}{\sigma} \leq \frac{90.0 - 110.0}{10.0}\right) = \Phi(-2.00) = \boxed{.0288}.$$

To get the mean μ_0 that corresponds to 1%, we use the fact that the normal 1% critical value is $z_{.01} = 2.326$ from Table 4.1 or the “ ∞ ” row of Table A.5 and solve for μ in

$$.0100 = \Phi(-z_{.01}) = \Phi(-2.326) = \Phi\left(\frac{90.0 - \mu_0}{10.0}\right) \implies -2.326 = \frac{90.0 - \mu_0}{10.0}.$$

to find $\mu_0 = (10.0)(2.326) + 90.0 = \boxed{113.3}$.

2. Suppose that the time X in minutes it takes a random student of Math 3070 to complete Problem 4.128 has the cdf $F(x)$. What fraction of students can complete the problem in ten minutes or less? What is the median time that a student takes to complete the problem? What is the mean time that a student takes to complete the problem?

$$F(x) = \begin{cases} 0, & \text{if } x < 3; \\ 1 - \frac{9}{x^2}, & \text{if } x \geq 3. \end{cases}$$

The probability that the problem is completed in ten or less minutes is

$$P(X \leq 10) = F(10) = 1 - \frac{9}{10^2} = \boxed{.91}.$$

The median $\tilde{\mu}$ is the solution of $.5 = F(\tilde{\mu})$ or

$$.5 = 1 - \frac{9}{\tilde{\mu}^2}$$

which gives $\tilde{\mu} = \boxed{3\sqrt{2}} = 4.243$ minutes.

To find the mean we need the pdf $f(x) = F'(x)$, or

$$f(x) = \begin{cases} 0, & \text{if } x < 3; \\ \frac{18}{x^3}, & \text{if } x \geq 3. \end{cases}$$

The mean time is then

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_3^{\infty} \frac{18x dx}{x^3} = \int_3^{\infty} \frac{18 dx}{x^2} = \left[-\frac{18}{x}\right]_3^{\infty} = 0 - \left(-\frac{18}{3}\right) = \boxed{6 \text{ minutes}}.$$

3. Kearns Ferns has received a large shipment of plants from the distributor. The shipment is acceptable if at most 10% of the plants are defective. As manager, you have to decide whether to accept or reject the shipment. You decide to select twenty plants randomly and accept the shipment only if the number of defective plants in the sample is at most four. What is the probability that you reject the shipment when the actual proportion of defectives is 10%? What is the probability that you accept the shipment even though the actual proportion of defectives is 30%?

Let X denote the number of defective plants in the sample of twenty. Then X is a binomial random variable $X \sim \text{Binomial}(20, p)$. The probability that the shipment is rejected even though the probability of a “success” (the plant is defective) is only $p = .1$ is from Table A.1

$$P(\text{reject shipment}) = P(X > 4) = 1 - P(X \leq 4) = 1 - B(4; 20, .1) = 1 - .957 = \boxed{.043}.$$

The probability that the shipment is accepted when the probability of a defective plant in the shipment is large, $p = .3$, is from Table A.1

$$P(\text{accept shipment}) = P(X \leq 4) = B(4; 20, .3) = \boxed{.238}.$$

4. Suppose that planes arrive at the Salt Lake Airport according to a Poisson process with rate of 20 per hour. What is the probability that exactly 25 arrive in one hour? What are the expected value and standard deviation of the number of planes that arrive in ten minutes? What is the probability that at least four planes arrive in a half hour?

We are given that the number of arrivals X is a Poisson random variable with rate $\mu = \alpha t$ where t is the time period measured in hours. Since $\alpha = 20$ arrivals per hour, for one hour, the rate constant is $\mu = 20 \cdot 1 = 20$. The probability that exactly 25 arrive in one hour is

$$p(x; \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!} \quad \text{so} \quad p(25; 20) = \frac{e^{-20} \cdot 20^{25}}{25!} = \boxed{0.0446}.$$

Or we may use Table A.2

$$P(X = 25) = P(X \leq 25) - P(X \leq 24) = P(25; 20) - P(24; 20) = .888 - .843 = .045.$$

In ten minutes, $t = \frac{1}{6}$ hours and $\mu = 20 \cdot \frac{1}{6} = \frac{10}{3}$. It follows that the expected number of arrivals in this time is $E(X) = V(X) = \mu = \boxed{\frac{10}{3}} = 3.333$ and so $\text{sd}(X) = \sqrt{\frac{10}{3}} = 1.826$.

In half an hour, $t = .5$ hours and $\mu = 20 \cdot (.5) = 10$. The probability that at least four arrive in this time is from Table A.2,

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - P(3; 10) = 1 - .010 = \boxed{.990}.$$

5. Suppose that there are 24,000 undergraduate students at the University of Utah of whom 6,000 are over 21 years of age. If 400 students were randomly chosen without replacement, what is the probability that more than 110 of them are over 21? [Give an exact expression but don't evaluate.] Use an approximation to compute the probability. Explain why your approximation is valid.

Let X be the number of over 21's in the sample. Since the choice of students is without replacement, X is a hypergeometric variable with $N = 24,000$ in the total population, $M = 6,000$ over 21's in that population, sample size is $n = 400$. The probability that more than 110 are over 21 is

$$P(X > 110) = \sum_{x=111}^{400} h(x; 400, 24,000, 6,000) = \boxed{\sum_{x=111}^{400} \frac{\binom{6,000}{x} \binom{18,000}{400-x}}{\binom{24,000}{400}}}$$

Because the sample to population ratio is $\frac{n}{N} = \frac{400}{24,000} = .0167 < .05$, this experiment may be analyzed as if it were binomial with $p = \frac{M}{N} = \frac{6,000}{24,000} = .25$ by the rule of thumb on p.116. Because $np = 400 \cdot .25 = 100$ and $nq = 400 \cdot .75 = 300$ are both at least ten, then the probability may be estimated using the normal approximation with continuity correction by the rule of thumb on p. 161. Thus

$$P(X > 110) = 1 - P(X \leq 110) \approx 1 - \Phi\left(\frac{110 + .5 - np}{\sqrt{npq}}\right) = 1 - \Phi\left(\frac{110.5 - 100}{\sqrt{75}}\right) = 1 - \Phi(1.2124)$$

Because $1.2124 = .76 \cdot 1.21 + .24 \cdot 1.22$ and $\Phi(1.21) = .8869$ and $\Phi(1.22) = .8888$, by interpolating Table A.1 we find that

$$\Phi(1.2124) \approx .76\Phi(1.22) + .24\Phi(1.21) = .76 \cdot .8888 + .24 \cdot .8869 = .8874$$

so that

$$P(X > 110) \approx 1 - \Phi(1.2124) = 1 - .8874 = \boxed{.1126}.$$