(1.) [p. 82, prob. 68] *A Los Angeles businesswoman makes frequent trips to Washington D.C.; 50% of the time she travels on airline #1, 30% of the time on airline #2, and the remaining 20% of the time on airline #3. For airline #1, flights are late into D.C. 30% of the time and late into L.A. 10% of the time. For airline #2, these percentages are 25% and 20%, whereas for airline #3, the percentages are 40% and 25%. If on a certain trip we learn that she arrived late at exactly one of the two destinations, what are the posterior probabilities of having flown on airlines #1, #2 and #3? Assume that the chance of late arrival in L.A. is unaffected by what happens on the flight to D.C.*

Let's solve the problem using Bayes' Theorem and the tree method following the hint. At the base node there are three branches corresponding to airline. Let A_i denote the event that the ith airline was used for both trips. At the tip of each branch, there are three second generation branches corresponding to the events B_j, that the flight was late j times, where $j \in \{0, 1, 2\}$. If L is the event that the flight was late into D.C. amd M the event that the flight was late into L.A., then we can use the independence of L and M to compute the conditional probabilities $P(B_j|A_i)$. We then use the tree, or what is equivalent, Bayes' Theorem, to compute the desired conditional probability $P(A_i|B_1)$, the probability that airline i was used, given that round trip was late on exactly one leg.

Note that the event that the flight was not late on either leg, or zero lates $B_0 = L' \cap M'$, that the flight was not late into D.C. and not late into L.A. The event that the trip was late on exactly one leg $B_1 = (L \cap M') \cup (L' \cap M)$, that it was late int D.C. but on time into L.A. or it was on time into D.C. and late into L.A. The event that there were two late legs $B_2 = L \cap M$, that the flight was late into D.C. and late into L.A. Thus, since conditional probabilities are probabilities and using the assumption that L ane M are independent for each airline, inserting the given information we compute

$$
\begin{aligned}
P(B_0|A_1) &= P(L' \cap M'|A_1) = P(L'|A_1)P(M'|A_1) = (1 - .3)(1 - .1) = .63;\\
P(B_0|A_2) &= P(L' \cap M'|A_2) = P(L'|A_2)P(M'|A_2) = (1 - .25)(1 - .2) = .6;\\
P(B_0|A_3) &= P(L' \cap M'|A_3) = P(L'|A_3)P(M'|A_3) = (1 - .4)(1 - .25) = .45;\\
P(B_1|A_1) &= P((L' \cap M) \cup (L \cap M')|A_1) = P(L' \cap M|A_1) + P(L \cap M'|A_1)\\
&= P(L'|A_1)P(M|A_1) + P(L|A_1)P(M'|A_1) = (1 - .3)(.1) + (.3)(1 - .1) = .34;\\
P(B_1|A_2) &= P((L' \cap M) \cup (L \cap M')|A_2) = P(L' \cap M|A_2) + P(L \cap M'|A_2)\\
&= P(L'|A_2)P(M|A_2) + P(L|A_2)P(M'|A_2) = (1 - .25)(.2) + (.25)(1 - .2) = .35;\\
P(B_1|A_3) &= P((L' \cap M) \cup (L \cap M')|A_3) = P(L' \cap M|A_3) + P(L \cap M'|A_3)\\
&= P(L'|A_1)P(M|A_3) + P(L|A_1)P(M'|A_3) = (1 - .4)(.25) + (.4)(1 - .25) = .45;\\
P(B_2|A_1) &= P(L \cap M|A_1) = P(L'|A_1)P(M'|A_1) = (.3)(.1) = .03;\\
P(B_2|A_2) &= P(L \cap M|A_2) = P(L'|A_2)P(M'|A_2) = (.25)(.2) = .05;\\
P(B_2|A_3) &= P(L \cap M|A_3) = P(L'|A_3)P(M'|A_3) = (.4)(.25) = .1.
\end{aligned}
$$

By the total probability formula

$$
\begin{aligned}
P(B_1) &= P(A_1)P(B_1|A_1) + P(A_2)P(B_1|A_2)P(A_3)P(B_1|A_3)\\
&= (.5)(.34) + (.3)(.35) + (.2)(.45) = .365.
\end{aligned}
$$

Thus, the posterior probabilities are

$$
\begin{aligned}
\big(P(A_1|B_1), P(A_2|B_1), P(A_3|B_1)\big) &= \left(\frac{P(A_1 \cap B_1)}{P(B_1)}, \frac{P(A_2 \cap B_1)}{P(B_1)}, \frac{P(A_3 \cap B_1)}{P(B_1)}\right) \\
&= \left(\frac{\big(P(A_1)P(B_1|A_1)}{P(B_1)}, \frac{P(A_2)P(B_1|A_2)}{P(B_1)}, \frac{P(A_3)P(B_1|A_3)}{P(B_1)}\right) \\
&= \left(\frac{(.5)(.34)}{.365}, \frac{(.3)(.35)}{.365}, \frac{(.2)(.45)}{.365}\right) \\
&= (0.466,\ 0.288,\ 0.247).
\end{aligned}
$$