Math 3070 § 1. **First Midterm Exam** Name: Solutions
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(1.) *The old "Ski Utah" license plates consisted of three numbers followed by three letters. How many different old "Ski Utah" license plates are possible? Why? An old "Ski Utah" license plate is chosen at random. What is the probability that no number and no letter appears more than once? Why?*

There are 10 different numbers and 26 different letters. We use the product rule. Thus the total number of license plates is the number of ways to choose the first number times the number of ways to choose the second number times the number of ways to choose the third number times the number of choices for the first letter times the number of choices for the second times the number of choices for the third, namely

$$n = 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = \boxed{17,576,000}.$$

Assuming plates are equally likely, the probability is the ratio of the number of plates with distinct numbers and distinct letters over number of plates. But there are only nine numbers distinct from the first and eight distinct from the first two and only 25 letters distinct from the first and 24 distinct from the first two. Thus,

$$P = \frac{\#\{\text{distinct no.s \& distinct letters}\}}{\#\{\text{plates}\}} = \frac{10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24}{10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26} = \boxed{0.639}$$

(2.) *A bag contains 26 scrabble tiles, each labeled by a different letter of the alphabet. You choose a subset of seven tiles randomly from the bag without replacement. You may leave your answers in terms of binomial coefficients $\binom{p}{q}$. How many different seven letter combinations can be selected from the bag? Why? Let $\mathcal{A}$ be the event that your selection has none of the vowels (A, E, I, O, U). What is the probability $P(\mathcal{A})$? Why? Let $\mathcal{B}$ be the event that your selection includes the letter 'X.' What is the probability $P(\mathcal{B})$? Why? What is the probability that your selection has no vowels or includes the 'X'? Why?*

We are choosing combinations of 26 tiles taken seven at a time thus there are $\binom{26}{7}$ such. Indeed, we have 26 choices for the first tile, 25 left for the second, 23 left for the third, 22 for the fourth and 21 for the fifth. But this enumeration includes all orderings, so we divide by 7!, the number of orderings of five tiles.

$$n = \boxed{\binom{26}{7}} = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{7!} = 657,800.$$

There are 21 non-vowels in the bag. The number of five tile subsets of these 21 is $\binom{21}{7}$. Thus, assuming each choice is equally likely, the probability is this number divided by the total number of choices from (a),

$$P(\mathcal{A}) = \boxed{\frac{\binom{21}{7}}{\binom{26}{7}}} = 0.177.$$

If 'X' is included, there are four remaining letters to be chosen from the 25 other letters, thus

$$P(\mathcal{B}) = \boxed{\frac{\binom{25}{6}}{\binom{26}{7}}} = 0.269.$$

The event that the selection has no vowels or includes the 'X' is $\mathcal{A} \cup \mathcal{B}$ so we use the union formula. To count the number of $\mathcal{A} \cap \mathcal{B}$, besides the 'X' we select the remaining six letters from the 20 consonants without 'X', hence

$$P(\mathcal{A}\cup\mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A}\cap\mathcal{B}) = \boxed{\frac{\binom{21}{7}}{\binom{26}{7}} + \frac{\binom{25}{6}}{\binom{26}{7}} - \frac{\binom{20}{6}}{\binom{26}{7}}} = 0.387.$$

(3.) *In a telephone survey of 1000 adults, responders were asked their opinion about the cost of college education. The respondents were classified according to whether they currently had a child in college and whether they thought the loan burden for most college students is too high, the right amount, or too low. The proportion in each category is shown in the probability table. Suppose one respondent is chosen at random from this group. What is the probability that the respondent has a child in college (event D)? What is the conditional probability that the respondent thinks that the loan burden is too high (event A) given that they have a child in college? Are events A and D independent? Explain.* Since events A, B, C are mutually exclusive and exhaustive, so

	Too High (A)	Right Amount (B)	Too Little (C)	Sum
Child in College (D)	.35	.08	.01	.44
No Child in College (D')	.25	.20	.11	.56
Sum	.60	.28	.12	1.00

$D = (D\cap A)\cup(D\cap B)\cup(D\cap C)$ is a disjoint union so we sum

$$P(D) = P(D\cap A) + P(D\cap B) + P(D\cap C) = .35 + .08 + .01 = \boxed{.44}.$$

Similarly $A = (D\cap A)\cup(D'\cap A)$ is a disjoint union so

$$P(A) = P(D\cap A) + P(D'\cap A) = .35 + .25 = .60.$$

By the conditional probability formula, the conditional probability that the respondent thinks that the loan burden is too high given that they have a child in college is

$$P(A|D) = \frac{P(A\cap D)}{P(D)} = \frac{.35}{.44} = \boxed{0.795}.$$

The events A and D are not independent, because, for example

$$.60 = P(A) \neq P(A|D) = .795.$$

(4.) *In 1997, the American Journal of Sports Medicine published a study of 810 women collegiate rugby players with two common knee injuries: MCL sprains and ACL tears. For backfield players, it was found that 39% had MCL sprains and 61% had ACL tears. For forwards, it was found that 33% had MCL sprains and 67% had ACL tears. Since a rugby team has seven backs and eight forwards, we assume that 47% of the players with knee injuries are backs and 53% are forwards. What is the probability that a that a rugby player selected at random from this group of players is a forward and has experienced an MCL sprain? What is the probability that a that a player has experienced an MCL sprain? Given that the player has an MCL sprain, what is the probability that the player is a forward?*

Let M be the event that the athlete has an MCL sprain and so M' is the event that the athlete has an ACL tear. Let B be the event that the player is a back, so B' is the event that the player is a forward. The probability that a player is a forward and has an MCL tear is using the product formula and the given information

$$P(B' \cap M) = P(B') \cdot P(M|B') = (.53)(.33) = \boxed{0.1749}$$

Since having an MCL sprain may be written as a disjoint union $M = (B \cap M) \cup (B' \cap M)$, by the total probability formula

$$\begin{aligned} P(M) &= P(B \cap M) + P(B' \cap M) \\ &= P(B)P(M|B) + P(B')P(M|B') \\ &= (.47)(.39) + (.53)(.33) = \boxed{0.3582} \end{aligned}$$

Given that the player has an MCL sprain, what is the probability that the player is a forward is using the definition of conditional probability,

$$P(B'|M) = \frac{P(B' \cap M)}{P(M)} = \frac{0.1749}{0.3582} = \boxed{.4483}$$

(5.) *Of the LED's manufactured by Lynndyl Electronics, the probability that a given LED works is p, where $0 < p < 1$. Four LED's are chosen at random. Assume that they function independently. Let $\mathcal{A}_i$ denote the event that the ith LED works, where $i \in \{1, 2, 3, 4\}$. Express the compound event "none of the LED's work" in terms of the $\mathcal{A}_i$'s. What is the probability that none of the LED's work? Be sure to explain where the independence assumption is used. Express the compound event "at least one LED works" in terms of the $\mathcal{A}_i$'s. What is the probability that at least one LED works? Express the compound event "exactly three LED's work" in terms of the $\mathcal{A}_i$'s. What is the probability that exactly three LED works?*

The event "none of the LED's work" may be rendered

$$N = \boxed{\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3' \cap \mathcal{A}_4'}.$$

Because the A_i are mutually independent, it follows that the A_i' are too, thus the probability of their intersection may be found by the product

$$P(N) = P(\mathcal{A}_1')P(\mathcal{A}_2')P(\mathcal{A}_3')P(\mathcal{A}_4) = \boxed{(1-p)^4}$$

The event "at least of the LED's work" may be rendered

$$L = \boxed{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4}$$

Note that the complementary event $L' = N$ is "none of the LED's work" so

$$P(L) = 1 - P(L') = 1 - P(N) = \boxed{1 - (1-p)^4}$$

The event $E =$ "exactly three LED's work" is

$$\boxed{(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4') \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3' \cap \mathcal{A}_4) \cup (\mathcal{A}_1 \cap \mathcal{A}_2' \cap \mathcal{A}_3 \cap \mathcal{A}_4) \cup (\mathcal{A}_1' \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4)}$$

As this is a disjoint union of four sets with equal probability, thus

$$P(E) = 4P(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4') = 4P(\mathcal{A}_1)P(\mathcal{A}_2)P(\mathcal{A}_3)P(\mathcal{A}_4') = \boxed{4p^3(1-p).}$$