Math 3080 § 1. **First Midterm Exam** Name: Solutions
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1. *In a study of the effect of shelf height in canned dog food sales, the number of cans sold per day of a single brand of dog food was observed. Three shelf heights were tested: knee level, waist level and eye level. The sales for each height was recorded for ten days. Fill in the blanks in the ANOVA table. (Explain your computations.) Do the data present evidence that there is a difference in sales for different shelf heights? State the model and the assumptions on the data. State the null and alternative hypotheses. State the test statistic and the rejection region for a level $\alpha = .05$ test. What is your conclusion?*

Source	df	SS	MS	F
Shelf Height			130	
Error				
Total		1060		

There are $I = 3$ levels of the treatment (Shelf Height) and $J = 10$ replicates per cell (number of days). Thus shelf height degrees of freedom is $I - 1 = \boxed{2}$, error degrees of freedom is $I(J-1) = 3 \cdot 9 = \boxed{27}$ and total degrees of freedom $IJ - 1 = 3 \cdot 10 - 1 = \boxed{29}$. Thus $SSTr = (I-1)MSTr = 2 \cdot 130 = \boxed{260}$. Using the partitioning of the sum of squares, $SSE = SST - SSTr = 1060 - 260 = \boxed{800}$. Hence $MSE = SSE/[I(J-1)] = 800/27 = \boxed{29.62963}$. Finally the f ratio is $f = MSTr/MSE = 130/29.62963 = \boxed{4.3875}$. We have completed the ANOVA table.

Source	df	SS	MS	f
Shelf Height	2	260	130	4 3875
Error	27	800	29.62963	
Total	29	1060		

The model is $x_{ij} = \mu + \alpha_i + \epsilon_{ij}$ where $\sum_{i=1}^{I} \alpha_i = 0$ and i.i.d. $\epsilon_{ij} \sim N(0, \sigma^2)$. The null and alternative hypotheses are

$$\mathcal{H}_0: \ \alpha_i = 0 \text{ for all } i. \qquad \text{vs.} \qquad \mathcal{H}_a: \ \alpha_i \neq 0 \text{ for some } i.$$

The test statistic is f. The null hypothesis is rejected f exceeds the critical $f > f(\alpha, I-1, I(J-1))$. For this problem, by Table A.9., The critical $f(.05, 2, 27) = 3.35$. Since the data has $f = 4.3875 > f_{\text{crit}}$, there is strong evidence to reject the null hypothesis: that there is a difference in dog food sales depending on shelf height.

2. *A canned data set from* **R**©, `npk`, in MASS, gives results about an experiment on the growth of peas. The garden was split into 6 blocks according to soil type. Four different fertilizers were randomly assigned to 1/70 acre plots in each block. The yield is in pounds per plot. Do the data present evidence that there is a difference in fertilizers? State the model and the assumptions on the data. State the null and alternative hypotheses. State the test statistic and the rejection region for a level $\alpha = .05$ test. What is the conclusion of the test? Use Tukey's procedure to identify differences in the true average yields for the four fertilizers. Use the method of underscoring to illustrate your conclusions. Write a sentence summarizing your results.

```
     Fertilizer
Block     1      2      3      4     mean
    1   46.8   62.8   49.5   57.0   54.025
    2   56.0   59.8   55.5   58.5   57.450
    3   62.8   69.5   55.0   55.8   60.775
    4   44.2   62.0   45.5   48.8   50.125
    5   51.5   52.0   48.8   49.8   50.525
    6   56.0   59.0   53.2   57.2   56.350
 mean   52.883 60.850 51.250 54.517 54.875

> a1 = aov(yield ~ Fertilizer + Block); anova(a1)
            Df Sum Sq Mean Sq F value   Pr(>F)
Fertilizer   3 317.62 105.873  7.3710 0.002911
Block        5 343.30  68.659  4.7801 0.008215
Residuals   15 215.45  14.363
```

The data is assumed to satisfy the linear model, that is

$$x_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \quad \text{where } \sum_{i=1}^{I} \alpha_i = 0,\ \sum_{j=1}^{J} \beta_j = 0 \text{ and i.i.d. } \epsilon_{ij} \sim N(0, \sigma^2).$$

The null and alternative hypotheses are

$$\mathcal{H}_{A0}: \ \alpha_i = 0 \text{ for all } i. \qquad \text{vs.} \qquad \mathcal{H}_{Aa}: \ \alpha_i \neq 0 \text{ for some } i.$$
$$\mathcal{H}_{B0}: \ \alpha_i = 0 \text{ for all } i. \qquad \text{vs.} \qquad \mathcal{H}_{Ba}: \ \alpha_i \neq 0 \text{ for some } i.$$

The test statistics are f_A and f_B. The null hypothesis $\mathcal{H}_{A0}$ is rejected if f_A exceeds the critical $f_A > f(\alpha, I-1, (I-1)(J-1))$. The null hypothesis $\mathcal{H}_{B0}$ is rejected if f_B exceeds the critical $f_A > f(\alpha, J-1, (I-1)(J-1))$. For this data, both p-values are far below $\alpha = .05$ so both are rejected: there is strong evidence that both the fertilizer and block effects are nonzero.

To see which fertilizers give significantly different yields, we order the means

$$\overline{x}_{3\cdot} = 51.25 < \overline{x}_{1\cdot} = 52.88 < \overline{x}_{4\cdot} = 54.52 < \overline{x}_{2\cdot} = 60.85$$

Means are significantly different is they exceed Tukey's Honest Significant differences

$$w = q(\alpha, I, (I-1)(J-1))\sqrt{\frac{MSE}{J}} = q(.05, 4, 15)\sqrt{\frac{14.363}{6}} = 6.31$$

where, by Table A.10, the critical value for the Studentized Range $q(.05, 4, 15) = 4.08$. Hence $\overline{x}_{2\cdot} - \overline{x}_{3\cdot} = 9.60$, $\overline{x}_{2\cdot} - \overline{x}_{1\cdot} = 7.97$ and $\overline{x}_{2\cdot} - \overline{x}_{4\cdot} = 6.33$ are significant and the other differences are not. The underscoring pattern is

```
Fertilizers            3        1          4          2
                     51.25    52.88      54.52      60.85
                     --------------------------
```

A summary sentence is "Fertilizer 2 gives a significantly higher yield than each of fertilizers 1,3 and 4 but fertilizers 1,3 and 4 do not differ significantly from each other."

3. *In a one factor random effects model*

$$x_{ij} = \mu + A_i + \epsilon_{ij} \qquad \text{where i.i.d. } A_i \sim N(0, \sigma_A^2) \text{ are independent of i.i.d. } \epsilon_{ij} \sim N(0, \sigma^2)$$

The following data shows the production of three operators chosen randomly from all operators, doing two trials on a particular machine. State the null and alternative hypotheses, the test statistic and rejection region. Give the estimate of σ^2 from this data. Give the estimate of σ_A^2 from this data.

		1	2	Mean
	1	156	164	160
Operator	2	163	173	168
	3	156	160	158

The null and alternative hypothes1s is

$$\mathcal{H}_0 : \ \sigma_A = 0 \qquad \text{vs.} \qquad \mathcal{H}_a : \ \sigma_A \neq 0.$$

The test statistics are $f_A = MSA/MSE$. The null hypothesis $\mathcal{H}_0$ is rejected if f_A exceeds the critical $f_A > f(\alpha, I-1, I(J-1))$.

In this problem, $I = 3$, $J = 2$, treatment d.f. is $I - 1 = 2$ and error d.f. is $I(J-1) = 3$. The average of the means is $\overline{x}.. = (160 + 168 + 158)/3 = 162$.

$$SST = \sum_i \sum_j (x_{ij} - \overline{x}..)^2 = 6^2 + 2^2 + 1^2 + 11^2 + 6^2 + 2^2 = 202$$
$$SSTr = \sum_i \sum_j (\overline{x}_{i\cdot} - \overline{x}..)^2 = 2(2^2 + 6^2 + 4^2) = 112$$
$$SSE = SST - SSTr = 202 - 112 = 90$$

Because $\mathbb{E}(MSE) = \sigma^2$, the estimator for $\widehat{\sigma^2} = MSE = SSE/[I(J-1)] = 90/3 = \boxed{30}$.
Because $\mathbb{E}(MSTr) = \sigma^2 + J\sigma_A^2$, the estimator for

$$\widehat{\sigma_A^2} = \frac{MSTr - MSE}{J} = \frac{SSTr/(I-1) - MSE}{J} = \frac{112/2 - 30}{2} = \boxed{13}.$$

4. *The extraction rates of a polymer are known to depend on reaction temperature and the amount of catalyst used. Here is output from* **R**©. *(From Walpole, Myers & Myers, 1998.) State the model and the assumptions on the data. State the null and alternative hypotheses. At the* .05 *level of significance, what conclusions do you draw? For each of the six diagnostic plots for this data on the next page, state what can be learned from the plot. What does the plot say about this data and about how well it satisfies the assumptions?*

```
               Catalyst
Temperature     0.5%       0.6%       0.7%       0.8%       0.9%
        50C    38  41     45  47     57  59     59  61     57  58
        60C    44  43     56  57     70  69     73  72     61  58
        70C    44  47     56  60     70  67     73  61     61  59
        80C    49  47     62  65     70  55     62  69     53  58

> az = aov(extraction.rate ~ temp * catalyst); summary(az);
                Df  Sum Sq Mean Sq F value    Pr(>F)
temp             3  430.48  143.49 10.8500 0.0001907
catalyst         4 2466.65  616.66 46.6285  7.28e-10
temp:catalyst   12  326.15   27.18  2.0551 0.0744559
Residuals       20  264.50   13.23
```

The data is assumed to satisfy

$$x_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij}$$

where $\sum_{i=1}^{I} \alpha_i = 0$, $\sum_{j=1}^{J} \beta_j = 0$, $\sum_{j=1}^{J} \gamma_{ij} = 0$ for each j, $\sum_{j=1}^{J} \gamma_{ij} = 0$ for each i and errors are i.i.d. $\epsilon_{ij} \sim N(0, \sigma^2)$. The null and alternative hypotheses are

$$\begin{aligned} \mathcal{H}_{A0}:\ \alpha_i = 0 \text{ for all } i. \quad &\text{vs.} \quad \mathcal{H}_{Aa}:\ \alpha_i \neq 0 \text{ for some } i.\\ \mathcal{H}_{B0}:\ \beta_j = 0 \text{ for all } j. \quad &\text{vs.} \quad \mathcal{H}_{Ba}:\ \beta_j \neq 0 \text{ for some } j.\\ \mathcal{H}_{AB0}:\ \gamma_{ij} = 0 \text{ for all } (i,j). \quad &\text{vs.} \quad \mathcal{H}_{ABa}:\ \gamma_{ij} \neq 0 \text{ for some } (i,j). \end{aligned}$$

The test statistics are f_A, f_B and f_{AB}. The null hypothesis $\mathcal{H}_{AB0}$ is rejected if f_{AB} exceeds the critical $f_{AB} > f(\alpha, (I-1)(J-1), IJ(K-1))$. If $\mathcal{H}_{AB0}$ is rejected the analysis normally stops. If not, the null hypothesis $\mathcal{H}_{A0}$ is rejected if f_A exceeds the critical $f_A > f(\alpha, I-1, IJ(K-1))$ or the null hypothesis $\mathcal{H}_{B0}$ is rejected if f_B exceeds the critical $f_A > f(\alpha, J-1, IJ(K-1))$. For this data, the p-value for rejecting $\mathcal{H}_{AB0}$ is .0745 which is close, but not rejected at the $\alpha = .05$ level. Both remaining p-values are far below $\alpha = .05$ so both are rejected: there is strong evidence that both the catalyst and temperature effects are nonzero.

Plot 1. shows that there seems to be some differences in temperature means and the variances in each temperature is pretty uniform, consistent with assumptions.

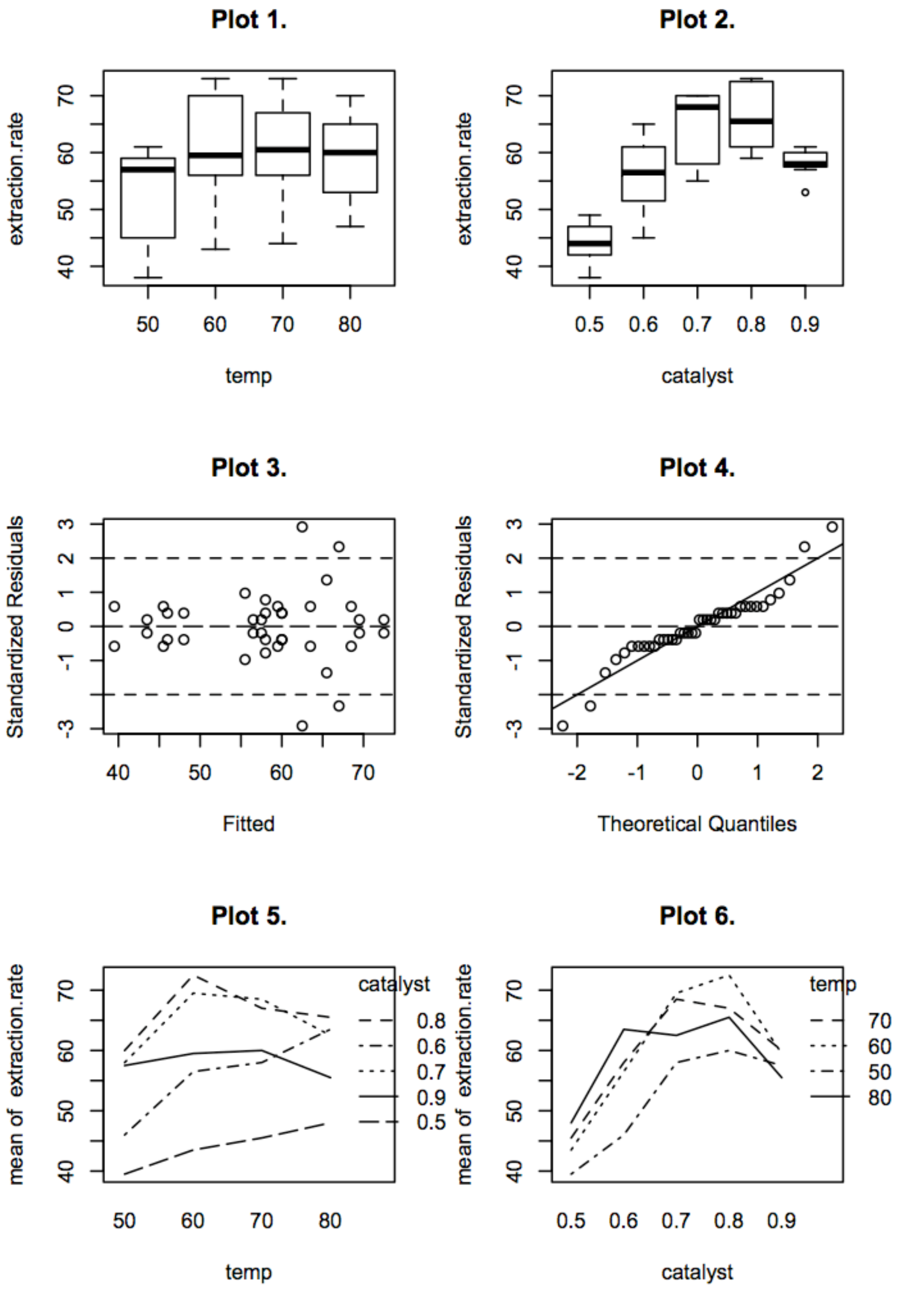
Plot 2. shows that there seems to be a significant differences in temperature means and quite different variances in catalysts, inconsistent with the assumptions.

Plot 3. shows a funnel shape: there seems to be some increase in spread of residuals with increase in fitted values, indicating that there may be some dependence of variance on value, contrary to the assumption that σ is uniform.

Plot 4. shows points follow somewhat of an "N" shape rather than agreeing with the 45^{deg} line. This suggests that the error distribution is heavier tailed than normal, contrary to the assumption that errors be normally distributed.

Plot 5. shows that there is some small interaction. The 0.9 and 0.5 catalyst curves cross at higher temperatures instead of being vertical translates, suggesting some γ_{ij} may not be zero. Indeed, we only marginally failed to reject $\mathcal{H}_{AB0}$.

Plot 6. shows the same, that there is some small interaction. The 80 temperature curve occasionally moves in different directions than the others, suggesting some γ_{ij} may not be zero.



5. *An article by Borges et al. in Communications in Soil Science and Plant Analysis, 2001, described an experiment in which pH levels of four alluvial soils were measured. Various levels of liming were applied to each soil. State the model and the assumptions on the data. Is the pH of Soil D significantly higher than the average of the others? State the null and alternative hypotheses, the test statistic and rejection region. What is your conclusion using* $\alpha = .01$*?*

```
     Liming Level
Soil    1    2    3    4    5   Mean
   A   5.8  5.9  6.1  6.5  7.1  6.30
   B   5.2  5.7  6.0  6.4  6.8  6.02
   C   5.5  6.0  6.2  6.7  7.0  6.28
   D   6.0  6.6  6.7  6.7  7.5  6.70

> a5 = aov(y ~ soil + lime); summary(a5)
            Df Sum Sq Mean Sq F value    Pr(>F)
soil         3  1.178 0.39267  18.335 8.903e-05
lime         4  5.047 1.26175  58.914 8.683e-08
Residuals   12  0.257 0.02142
```

The data is assumed to satisfy the linear model, that is

$$x_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \quad \text{where} \sum_{i=1}^{I} \alpha_i = 0, \sum_{j=1}^{J} \beta_j = 0 \text{ and i.i.d. } \epsilon_{ij} \sim N(0, \sigma^2).$$

(You were not asked for an analysis of the results of ANOVA, but here it is. The null and alternative hypotheses are

$$\begin{aligned} \mathcal{H}_{A0}:\ \alpha_i = 0 \text{ for all } i. \qquad &\text{vs.} \qquad \mathcal{H}_{Aa}:\ \alpha_i \neq 0 \text{ for some } i. \\ \mathcal{H}_{B0}:\ \alpha_i = 0 \text{ for all } i. \qquad &\text{vs.} \qquad \mathcal{H}_{Ba}:\ \alpha_i \neq 0 \text{ for some } i. \end{aligned}$$

The test statistics are f_A and f_B. The null hypothesis $\mathcal{H}_{A0}$ is rejected if f_A exceeds the critical $f_A > f(\alpha, I-1, (I-1)(J-1))$. The null hypothesis $\mathcal{H}_{B0}$ is rejected if f_B exceeds the critical $f_A > f(\alpha, J-1, (I-1)(J-1))$. For this data, both p-values are far below $\alpha = .01$ so both are rejected: there is strong evidence that both the soil and liming level effects on pH are nonzero.)

To answer the question, let $\mu_i = \mu + \alpha_i$ be the soil means. There are $I = 4$ soils, $J = 5$ lime levels and one replicate per cell. We test whether the data shows that the contrast is negative

$$\theta = \frac{\mu_1 + \mu_2 + \mu_3}{3} - \mu_4$$

where $c_1 = c_2 = c_3 = 1/3$ and $c_4 = -1$. Its estimator is

$$\hat{\theta} = \frac{\overline{x}_{1\cdot} + \overline{x}_{2\cdot} + \overline{x}_{3\cdot}}{3} - \overline{x}_{4\cdot} = \frac{6.30 + 6.02 + 6.28}{3} - 6.70 = -.50$$

The null and alternative hypotheses are

$$\mathcal{H}_0:\ \theta < 0 \qquad \text{vs.} \qquad \mathcal{H}_a:\ \theta \geq 0$$

The test statistic is the standardized estimator

$$t = \frac{\hat{\theta}}{\sqrt{\frac{MSE}{J}\sum_{i=1}^{I} c_i^2}} = \frac{-.50}{\sqrt{\frac{.02142}{5}\left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3^2} + (-1)^2\right)}} = -6.616$$

The null hypothesis is rejected t falls below the critical $t < -t(\alpha, (I-1)(J-1))$. For this problem, by Table A.5., the critical $t(.01, 12) = 2.681$. Since the data has $t = -6.616 < -t_{\text{crit}}$, there is strong evidence to reject the null hypothesis: the fourth soil has greater pH than the average of the other soils.