Math 3210 § 2. **First Midterm Exam** Name: Solutions
Treibergs January 22, 2020

1. *Prove that for every natural number n,* $$\sum_{k=1}^{n} k(k!) = (n+1)! - 1.$$

Proof by induction.

Base Case. *Let $n=1$. Then the left side equals $\sum_{k=1}^{n} k(k!) = \sum_{k=1}^{1} k(k!) = 1 \cdot 1! = 1$. The right side is $(n+1)!-1 = 2!-1 = 1$. Since these are equal, the base case holds.*

Induction Case. *Assume for some $n \in \mathbb{N}$ that* $\displaystyle\sum_{k=1}^{n} k(k!) = (n+1)! - 1$. *Then for $n+1$, by applying the induction hypothesis,*

$$\begin{aligned}\sum_{k=1}^{n+1} k(k!) &= \sum_{k=1}^{n} k(k!) + (n+1)(n+1)! \\ &= (n+1)! - 1 + (n+1)(n+1)! \\ &= (n+1)![1+n+1] - 1 \\ &= (n+1)![n+2] - 1 \\ &= (n+2)! - 1 \\ &= [(n+1)+1]! - 1,\end{aligned}$$

so that the equation holds with $n+1$ establishing the induction step. As the base case and induction cases hold, by induction, $\displaystyle\sum_{k=1}^{n} k(k!) = (n+1)! - 1$ *holds for all $n \in \mathbb{N}$.*

2. *Assume that the real numbers a, b, c and d satisfy $ad - bc \neq 0$. Let $f(x) = \dfrac{ax+b}{cx+d}$. Determine the natural domain $X = \{x \in \mathbf{R} : f(x)$ is defined.$\}$. Does it depend on a, b, c or d? Define: $f : X \to \mathbf{R}$ is one-to-one. Define: $f : X \to \mathbf{R}$ is onto. Prove that $f : X \to \mathbf{R}$ is one-to-one. Find $f(X)$. Is f onto? Why or why not?*

The natural domain is $X = \{x \in \mathbf{R} : cx + d \neq 0\}$. In case $c = 0$ then $ad - bc = 0$ implies $ad \neq 0$ so $a \neq 0$ and $d \neq 0$. So if $c = 0$ then $X = \mathbf{R}$. Otherwise $X = \mathbf{R} \setminus \left\{-\frac{d}{c}\right\}$.

$f : X \to \mathbf{R}$ is *one-to-one* if whenever for $x, y \in X$ such that $x \neq y$ then $f(x) \neq f(y)$. f is *onto* if $f(X) = \mathbf{R}$, or in other words, for every $y \in \mathbf{R}$, there is $x \in X$ such that $f(x) = y$.

To show f is one-to-one, suppose there are $x, y \in X$ such that $f(x) = f(y)$. This implies

$$\frac{ax+b}{cx+d} = \frac{ay+b}{cy+d}.$$

Hence $(ax+b)(cy+d) = (ay+b)(cx+d)$ or after expanding,

$$acxy + adx + bcy + bd = acxy + ady + bcx + bd.$$

Simplifying,

$$(ad-bc)x = (ad-bc)y.$$

But since $ad - bc \neq 0$ we may divide both sides by it to deduce $x = y$, namely, f is one-to-one on X.

$f : X \to \mathbf{R}$ is onto if $c = 0$. Otherwise f is not onto. To see it, choose $y \in \mathbf{R}$ and try to solve for $x \in X$ so that $f(x) = y$. In case $c = 0$ which implies $a \neq 0$ and $d \neq 0$ we solve

$$y = f(x) = \frac{ax+b}{d}$$

to get

$$x = \frac{dy-b}{a}.$$

Thus in this case, f is onto and in this case, $f(X) = \mathbf{R}$. (After all, f is in this case just a linear function with nonvanishing x coefficient.)

In case $c \neq 0$, then f is not onto. In trying to solve for some $y \in \mathbf{R}$ we see that

$$y = f(x) = \frac{ax+b}{cx+d}$$

implies

$$(cx+d)y = ax+b$$

so

$$(cy-a)x = b - dy.$$

Thus, if soluble this gives

$$x = \frac{b-dy}{cy-a}.$$

This equation has no solution if $cy = a$, *i.e.*, $f(X)$ misses the point $y = \frac{a}{c}$ so $f(X) = \mathbf{R}\backslash\{\frac{a}{c}\}$. Thus f is not onto.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Let X be a set and E_α be subsets of X for all $\alpha \in A$. Then* $X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A}(X\backslash E_\alpha)$.

   TRUE. We show $x \in X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right)$ if and only if $x \in \bigcap_{\alpha\in A}(X\backslash E_\alpha)$. Indeed, by deMorgan's Law and the distributivity for logical statements,

$$\begin{aligned}
x \in X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) &\iff [x \in X] \wedge \left[x \notin \left(\bigcup_{\alpha\in A} E_\alpha\right)\right] \\
&\iff [x \in X] \wedge \left[\sim \left\{x \in \left(\bigcup_{\alpha\in A} E_\alpha\right)\right\}\right] \\
&\iff [x \in X] \wedge [\sim \{(\exists \alpha \in A) x \in E_\alpha\}] \\
&\iff [x \in X] \wedge [(\forall \alpha \in A) \sim \{x \in E_\alpha\}] \\
&\iff [x \in X] \wedge [(\forall \alpha \in A) x \notin E_\alpha] \\
&\iff (\forall \alpha \in A)\,([x \in X] \wedge [x \notin E_\alpha]) \\
&\iff (\forall \alpha \in A)\,(x \in X\backslash E_\alpha) \\
&\iff x \in \bigcap_{\alpha\in A}(X\backslash E_\alpha).
\end{aligned}$$

   (b) STATEMENT. *If $f : A \to B$ then $f^{-1}(f(E)) = E$ for every subset $E \subset A$.*

   FALSE. Let $A = B = \mathbf{R}$ and $f(x) = x^2$. Choosing $E = [2, 3)$ we see that $f(E) = [4, 9)$ and $f^{-1}(f(E)) = (-3, 2] \cup [2, 3) \neq E$.

(c) *Let $f : A \to B$ be a function. Then $f(E \cap G) = f(E) \cap f(G)$ for all subsets E and G of A.*

FALSE. Let $A = B = \mathbf{R}$ and $f(x) = x^2$. For $E = [1, 2]$ and $G = (-2, -1)$ we have $E \cap G = \emptyset$ so $f(E \cap G) = \emptyset$ but $f(E) = [1, 4]$ and $f(G) = (1, 4)$ so $F(E) \cap f(E) = (1, 4) \neq f(E \cap G)$.

4. (a) *Let P and Q be logical statements. Prove that $(P \vee Q) \implies (P \wedge Q)$ is equivalent to $P \iff Q$.*

We prove this using truth tables. The two composite statments have the same truth values, so are equivalent.

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \implies (P \wedge Q)$	$P \iff Q$
T	T	T	T	T	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	F	F	T	T

(b) *Recall the Peano Axioms for the natural numbers $\mathbb{N}$:*

[N1.] There is an element $1 \in \mathbb{N}$.

[N2.] For each $n \in \mathbb{N}$ there is a successor element $s(x) \in \mathbb{N}$.

[N3.] 1 is not the successor of an element of $\mathbb{N}$.

[N4.] If two elements of $\mathbb{N}$ have the same successor, then they are equal.

[N5.] If a subset $A \subset \mathbb{N}$ contans 1 and is closed under succession (meaning $s(n) \in A$ whenever $n \in A$), then $A = \mathbb{N}$.

The Peano Axioms don't define addition nor multiplication. For $m, k \in \mathbb{N}$, how are $m + k$ and mk defined? How are the Peano Axioms used in these definitions?

The Peano Axioms imply that these formulae may be uniquely defined recursively. Choose $n \in \mathbb{N}$.

For addition, we define the sequence $x_k = n + k$ for $k \in \mathbb{N}$. The initial term is $x_1 = n + 1 = s(n)$, and then for $k \in \mathbb{N}$, $x_{k+1} = n + (k + 1) = s(n + k) = s(x_k)$, where $s(n)$ is the successor function in $\mathbb{N}$.

Using addition just defined, we may now define multiplication analgously. We define the sequence $y_k = n \cdot k$ for $k \in \mathbb{N}$. We take as initial term $y_1 = n \cdot 1 = n$, and then for $k \in \mathbb{N}$, $y_{k+1} = n \cdot (k + 1) = (n \cdot k) + n = y_k + n$.

One then shows that the arithmetic properties of $(\mathbb{N}, +, \cdot)$ hold, using induction and the definitions just given. For example, the associative property $(n+(m+k) = (n+m)+k$ for all $m, n, k \in \mathbb{N})$ and commutative property ($n + m = m + n$ for all $m, n \in \mathbb{N}$) of addition are argued in the text. There are analgous properties for multiplication and distributivity.

5. *Let* $\mathbb{N}$ *denote the natural numbers. Let* $E \subset \mathbf{R}$ *be a set of real numbers given by*

$$E = \{x \in \mathbf{R} : \quad (\forall n \in \mathbb{N}) \quad (\exists m \in \mathbb{N}) \quad x < n \implies x > m \quad \}.$$

Express E *in terms of intervals and and prove your expression equals* E.

Using $P \implies Q$ is equivalent to $(\sim P) \vee Q$, we see that

$$\begin{aligned} E &= \{x \in \mathbf{R} : (\forall n \in \mathbb{N})(\exists m \in \mathbb{N}) \quad x < n \implies x > m\} \\ &= \{x \in \mathbf{R} : (\forall n \in \mathbb{N})(\exists m \in \mathbb{N}) \quad (x \geq n) \vee (x > m)\} \\ &= \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} [n, \infty) \cup (m, \infty) \\ &= \bigcap_{n \in \mathbb{N}} \begin{cases} [1, \infty), & \text{if } n = 1; \\ (1, \infty), & \text{if } n > 1. \end{cases} \\ &= (1, \infty). \end{aligned}$$

To prove $E = (1, \infty)$, we show that $(1, \infty) \subset E$ and $E \subset (1, \infty)$.

Suppose $x \in (1, \infty)$ to show $x \in E$. Thus $x > 1$. Choose $n \in \mathbb{N}$ and let $m = 1$. Then $x > m$ is true, therefore $x < n \implies x > m$ is true whether or not $x < n$ is true. Thus $x \in E$.

We show the contrapositive: if $x \notin (1, \infty)$ which is equivalent to $x \leq 1$ then $x \notin E$. But $x \notin E$ is equivalent to $\sim (\forall n \in \mathbb{N})(\exists m \in \mathbb{N})[x < n \implies x > m]$ is equivalent to $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})[(x < n) \wedge (x \leq m)]$. By taking $n = 2$ and then for any $m \in \mathbb{N}$, $x < n \wedge x \leq m$ is true since $x \leq 1$. Thus $x \notin E$.