Math 3210 § 2. **Second Midterm Exam** Name: Solutions

Treibergs February 19, 2020

1. *Let $A = \{x \in \mathbb{Q} : x^2 - 2x < 8\}$, where $\mathbb{Q}$ denotes the rational numbers. Define: M is the least upper bound of A. Show that A is nonempty. Show that A is bounded above. Find the least upper bound of A and prove your result.*

   Let A be a nonempty subset of the real numbers which is bounded above. Then the real number M is the *least upper bound of* A if (1) it is an upper bound $(\forall a \in A)(a \le M)$, and (2) it is the least of all upper bounds, that is, no smaller number is an upper bound $(\forall x < M)(\exists a \in A)(x < a)$.

   Let $f(x) = x^2 - 2x - 8 = (x-4)(x+2)$. Then the condition to be in A is that x be rational and $f(x) < 0$. A is nonempty because the number $0 \in A$: 0 is rational and $f(0) = -8 < 0$.

   4 is an upper bound for A. If $x > 4$ we show $x \notin A$ so that whatever is left in A is at most four. If $x > 4$ then $x - 4 > 0$ and $x + 2 > 0$ so their product $f(x) > 0$, thus $x \notin A$.

   We claim that 4 is also the least upper bound. We showed it is an upper bound. To show there is no smaller upper bound, suppose $x < 4$. Then $\max(-2, x) < 4$. By the density of rationals, there is $a \in \mathbb{Q}$ such that $\max(-2, x) < a < 4$. Then $a - 4 < 0$ and $a + 2 > 0$ so their product $f(a) < 0$ so $a \in A$. Thus there exists $a \in A$ such that $x < a$, thus x is not a lower bound.

2. *Recall the axioms of a field $(\mathcal{F}, +, \times)$. For any $x, y, z \in \mathcal{F}$,*

[A1.]	(Commutativity of Addition)	$x + y = y + x$.
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z$.
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\,(\forall\, t \in \mathcal{F})\ 0 + t = t$.
[A4.]	(Additive Inverse)	$(\exists -x \in \mathcal{F})\ x + (-x) = 0$.
[M1.]	(Commutativity of Multiplication)	$xy = yx$.
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z$.
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})\ 1 \ne 0$ and $(\forall\, t \in \mathcal{F})\ 1t = t$.
[M4.]	(Multiplicative Inverse)	If $x \ne 0$ then $(\exists\, x^{-1} \in \mathcal{F})\ (x^{-1})x = 1$.
[D.]	(Distributivity)	$x(y + z) = xy + xz$.

   Using only the field axioms, show that for any $a, b, c \in \mathcal{F}$ such that $a \ne 0$ there is at most one solution x to the equation

$$a(x + b) = c.$$

   Justify every step of your argument using just the axioms listed here.

   Suppose there exist two solutions x and y. Since they are both solutions, they satisfy $a(x + b) = c$ and $a(y + b) = c$.

$$
\begin{aligned}
a(x+b) &= a(y+b) && \text{Both equal } c.\\
a^{-1}[\,a(x+b)] &= a^{-1}[\,a(y+b)] && a \neq 0 \text{ so there is } a^{-1} \text{ by M4. Pre-multiply by } a^{-1}.\\
[\,a^{-1}a](x+b) &= [\,a^{-1}a](y+b) && \text{M2.}\\
[\,aa^{-1}](x+b) &= [\,aa^{-1}](y+b) && \text{M1.}\\
1(x+b) &= 1(y+b) && \text{M4.}\\
x+b &= y+b && \text{M3.}\\
(x+b)+(-b) &= (y+b)+(-b) && \text{By A4 there is } -b. \text{ Post-add } -b.\\
x+[b+(-b)] &= y+[b+(-b)] && \text{A2.}\\
x+0 &= y+0 && \text{A4.}\\
0+x &= 0+y && \text{A1.}\\
x &= y && \text{A3.}
\end{aligned}
$$

We have shown $x = y$, hence all solutions have to be the same.

Not asked in this problem is whether there exist any solutions. A formula for the solution may be found by solving for x, or by guessing x and checking that it solves the problem.

$$
\begin{aligned}
a(x+b) &= c && \text{The equation.}\\
a^{-1}[\,a(x+b)] &= a^{-1}c && a \neq 0 \text{ so there is } a^{-1} \text{ by M4. Pre-multiply by } a^{-1}.\\
[\,a^{-1}a](x+b) &= a^{-1}c && \text{M2.}\\
[\,aa^{-1}](x+b) &= a^{-1}c && \text{M1.}\\
1(x+b) &= a^{-1}c && \text{M4.}\\
x+b &= a^{-1}c && \text{M3.}\\
(x+b)+(-b) &= (a^{-1}c)+(-b) && \text{By A4 there is } -b. \text{ Post-add } -b.\\
x+[b+(-b)] &= (a^{-1}c)+(-b) && \text{A2.}\\
x+0 &= (a^{-1}c)+(-b) && \text{A4.}\\
0+x &= (a^{-1}c)+(-b) && \text{A1.}\\
x &= (a^{-1}c)+(-b) && \text{A3.}
\end{aligned}
$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement. *In an ordered field, if $xz \ge yz$ and $z > 0$ then $x \ge y$.*
   True. Since $z > 0$ we have $z \neq 0$ so z^{-1} exists and $z \ge 0$ implies $z^{-1} \ge 0$. (If not, $z^{-1} < 0$ so multiplying by $z \ge 0$ gives $1 = zz^{-1} = (z^{-1})z \le 0z^{-1} = 0$ contrary to $1 > 0$.) Hence the inequality is preserved upon multiplying by z^{-1}. It gives $(xz)z^{-1} \ge (yz)z^{-1}$ which implies $x = 1x = x1 = x(zz^{-1}) = (xz)z^{-1} \ge (yz)z^{-1} = y(zz^{-1}) = y1 = 1y = y$.

   (b) Statement. *Let $\{x_n\}$ be a convergent sequence such that every x_n is irrational. Then the limit $\lim\limits_{n\to\infty} x_n$ is irrational.*
   False. Let $x_n = \dfrac{\sqrt{2}}{n}$. Then x_n is irrational as it is the product of rational $\frac{1}{n}$ and irrational $\sqrt{2}$, but $x_n \to 0$ as $n \to \infty$, where the limit, 0, is rational.

(c) STATEMENT. *Let f and g be two real valued functions defined for all reals such that* $\sup_{\mathbf{R}} f = \sup_{\mathbf{R}} g = 1$. *Then* $\sup_{\mathbf{R}}(f+g) = 2$.

FALSE. Let $f(x) = \begin{cases} \sin x, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0. \end{cases}$ and $g(x) = \begin{cases} 0, & \text{if } x \ge 0; \\ \sin x, & \text{if } x < 0. \end{cases}$. Then $f(\mathbf{R}) = g(\mathbf{R}) = [-1,1]$ so $\sup_{\mathbf{R}} f = \sup_{\mathbf{R}} g = 1$. But $(f+g)(x) = \sin x$ so $(f+g)(\mathbf{R}) = [-1,1]$ and $\sup_{\mathbf{R}}(f+g) = 1 \ne 2$.

4. *Recall that the rational numbers are defined to be the set of equivalence classes $\mathbb{Q} = S/\sim$ where $S = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \ne 0 \right\}$ is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction $\frac{a}{b} \sim \frac{c}{d}$ iff $ad = bc$. We denote the equivalence class, the "fraction," $\left[\frac{a}{b}\right]$ to distinguish it from a symbol from S. Given fractions $x, y \in \mathbb{Q}$, how should addition $x + y$ and multiplication be xy defined to make $\mathbb{Q}$ a field? You don't need to check that these are well defined nor that the axioms of a field are satisfied. Suppose we wish to define the function $f : \mathbb{Q} \to \mathbb{Q}$ by $f\left(\left[\frac{a}{b}\right]\right) = \left[\frac{a^2}{a^2+b^2}\right]$.*

Is f well defined? Why or why not? State the Completeness Axiom for an ordered field $\mathcal{F}$. Do the rational numbers $\mathbb{Q}$ satisfy the Completeness Axiom? Why or why not?

Addition and multiplication are defined for arbitrary $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$ by

$$\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right], \qquad \left[\frac{a}{b}\right]\left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right].$$

One then checks this addition and multiplication are well defined and with these, $\mathbb{Q}$ satisfies the field axioms.

To show that f is well defined we need to show that if $\frac{a}{b} \sim \frac{c}{d}$ then $f\left(\left[\frac{a}{b}\right]\right) = f\left(\left[\frac{c}{d}\right]\right)$ which is the same as $\frac{a^2}{a^2+b^2} \sim \frac{c^2}{c^2+d^2}$. But $\frac{a}{b} \sim \frac{c}{d}$ holds if $ad = bc$. Now using this we see that $(c^2+d^2)a^2 = a^2c^2 + a^2d^2 = a^2c^2 + b^2c^2 = (a^2+b^2)c^2$ which says $\frac{a^2}{a^2+b^2} \sim \frac{c^2}{c^2+d^2}$.

The ordered field $\mathcal{F}$ satisfies the completeness axiom if every nonempty set of $\mathcal{F}$ which is bounded above has a least upper bound in $\mathcal{F}$.

The rationals are not complete. The set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above, (say by 3 since if $x > 3$ then $x^2 > 9 > 2$ so $x \notin A$, hence members of A are at most 3). The least upper bound would have to be $\sqrt{2}$, but $\sqrt{2}$ is not rational.

In fact we showed that if $q > 0$ is a rational upper bound for A so $q^2 > 2$ then $\tilde{q} = 1/q + q/2$ is rational, $0 < \tilde{q} < q$ but $(\tilde{q})^2 > 2$ so $\tilde{q}$ is a strictly smaller upper bound for A. Similarly, if $r > 0$ is rational such that $r^2 < 2$ then $\tilde{r} = 4r/(2+r^2)$ is rational, $(\tilde{r})^2 < 2$ and $r < \tilde{r}$ so that for any $r \in A$ there is a strictly greater $\tilde{r} \in A$. Thus the positive least upper bound must be smaller that any rational such that $q^2 > 2$ and larger than any rational such that $r^2 < 2$. Hence the least upper bound would satisfy $x^2 = 2$, but there is no such rational.)

5. *Let* $\{x_n\}$ *be a real sequence and* L *a real number. Define:* $L = \lim_{n\to\infty} x_n$*. Using just your definition, determine whether the limit* $L = \lim_{n\to\infty} \frac{n^2+n}{n^2-7}$ *exists and prove your answer.*

The sequence is said to tend to a limit, $L = \lim_{n\to\infty} x_n$, if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|x_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

We claim $1 = \lim_{n\to\infty} \frac{n^2+n}{n^2-7}$. To prove it, choose $\varepsilon > 0$. Let $N = \frac{16}{\varepsilon} + 4$. For any $n \in \mathbb{N}$ such that $n > N$ we have $n > 4$ so $n^2 - 7 > 0$, $7n > 7$ and $\frac{1}{2}n^2 > 7$. Thus

$$\begin{aligned} |x_n - L| &= \left| \frac{n^2+n}{n^2-7} - 1 \right| = \left| \frac{n^2+n}{n^2-7} - \frac{n^2-7}{n^2-7} \right| = \left| \frac{n+7}{n^2-7} \right| = \frac{n+7}{n^2-7} \\ &\le \frac{n+7n}{n^2 - \frac{1}{2}n^2} = \frac{8n}{\frac{1}{2}n^2} = \frac{16}{n} < \frac{16}{N} = \frac{16}{\frac{16}{\varepsilon}+4} = \frac{16\varepsilon}{16+4\varepsilon} < \frac{16\varepsilon}{16} = \varepsilon. \end{aligned}$$