Math 3210 § 2.	**Final Exam**	Name:____________________
Treibergs		April 28, 2020

This is an open book exam. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.______	/21
2.______	/22
3.______	/21
4.______	/22
5.______	/22
6.______	/21
7.______	/22
8.______	/22
9.______	/20
Total______	/150

1. (a) [10] Let $n \ge 1$ and $0 < x < y$. Show that $nx^{n-1}(y-x) \le y^n - x^n \le ny^{n-1}(y-x)$.

   (b) [11] Determine whether the improper integral exists.

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x^2)}$$

2. (a) [3] Let $f : (a,b) \to \mathbf{R}$. Define the *infimum of* f, $\inf\limits_{x\in(a,b)} f(x)$.

   (b) [19] Find $\inf\limits_{x\in(0,1)} \dfrac{1}{x+1}$ and prove your result.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] If $x, y > 0$ then $\log(xy) = \log x + \log y$.

   TRUE: ◯ FALSE: ◯

   (b) [7] If $f, g : [0,1] \to \mathbb{R}$ are bounded functions such that, fg is integrable on $[0,1]$, then at least one of f or g is integrable on $[0,1]$.

   TRUE: ◯ FALSE: ◯

   (c) [7] Let $f : \mathbf{R} \to \mathbf{R}$. Suppose both limits $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ exist. Then f is continuous at 0.

   TRUE: ◯ FALSE: ◯

4. (a) [3] Define what it means for f to be *integrable* on $[a,b]$ and what the *Riemann integral* of f on $[a,b]$ is.

(b) [19] Let $I=[0,1]$ and $f(x)=\begin{cases} x, & \text{if } x\in\mathbb{Q};\\ 0, & \text{in } x\notin\mathbb{Q}.\end{cases}$ where $\mathbb{Q}$ is the rational numbers.

Find $\underline{\int}_I f(x)\,dx$ and $\overline{\int}_I f(x)\,dx$ and explain. What do they tell you about f?

5. For $n \in \mathbf{N}$, let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions.

   (a) [3] Define: the sequence $\{a_n\}$ is a *Cauchy Sequence.*

   (b) [9] Prove using only the definition (a) that if $\{a_n\}$ is a Cauchy Sequence, then $\{a_n\}$ is bounded.

   (c) [10] Prove using only the definition (a) and the result (b) that if $\{a_n\}$ and $\{b_n\}$ are Cauchy Sequences, then $\{a_n b_n\}$ is a Cauchy Sequences.

6. (a) [3] Let $f, f_k : \mathbb{R} \to \mathbf{R}$. Define: $f(x) = \sum_{k=1}^{\infty} f_k(x)$ *converges uniformly* on $\mathbf{R}$.

(b) [18] Determine whether the series of functions $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly, where

$$f_k(x) = \begin{cases} \dfrac{x(k-x)}{k^4}, & \text{if } 0 \le x \le k; \\ 0, & \text{otherwise.} \end{cases}$$

7. Let $f(x) = \dfrac{1}{x+1}$, $a > -1$ and n be an integer. For each part, determine the limit and explain in sufficient detail to justify why your limit exists.

(a) [5] $\displaystyle\lim_{n\to\infty} f\left(a + \frac{1}{n}\right) =$

(b) [5] $\displaystyle\lim_{n\to\infty} n \cdot \left[f\left(a + \frac{1}{n}\right) - f(a)\right] =$

(c) [6] $\displaystyle\lim_{n\to\infty} n \cdot \int_a^{a+1/n} f(t)\,dt =$

(d) [6] $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(a + \frac{k}{n}\right) =.$

8. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Complete the statement of the theorem. [Of several possible answers, select the one you prefer for part (b).]

   Theorem. *The bounded function f is integrable on $[a, b]$ if and only if*

   (b) [19] Using only the theorem in (b), show that $f(x) = \sqrt{x(1-x)}$ is integrable on $[0, 1]$.

9. For each infinite series, determine whether the series is absolutely convergent, convergent of divergent.

(a) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \log\left(\frac{k+1}{k}\right).$

Absolutely Conv.: ◯	Conditionally Conv.: ◯	Divergent: ◯

(b) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{2^k}{3^k+1}.$

Absolutely Conv.: ◯	Conditionally Conv.: ◯	Divergent: ◯

(c) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{\log k}{\log(k^2+k+1)}.$

Absolutely Conv.: ◯	Conditionally Conv.: ◯	Divergent: ◯

(d) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k^k}{(2k+1)!}.$

Absolutely Conv.: ◯	Conditionally Conv.: ◯	Divergent: ◯

(e) [4] $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k(k+2)}{(k+1)(k+3)(k+5)}.$

Absolutely Conv.: ◯	Conditionally Conv.: ◯	Divergent: ◯