Math 3210 § 3. **Second Midterm Exam** Name: Solutions
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1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a real valued function defined on the reals. Define: $M = \sup_{\mathbf{R}} f$. Let $g(x) = \dfrac{x^2}{1+x^2}$. Find $M = \sup_{\mathbf{R}} g$ and prove your result.*

   Let $f : \mathbf{R} \to \mathbf{R}$ be a real function and $M \in \mathbf{R}$. If f is not bounded above we say $\sup_{\mathbf{R}} f = \infty$. If f is bounded above then we say $\sup_{\mathbf{R}} f = M$ where M is a real number such that (1) $f(x) \le M$ for all $x \in \mathbf{R}$ and (2) for every $s < M$ there is an $x \in \mathbf{R}$ such that $f(x) > s$.

   We claim $\sup_{\mathbf{R}} \dfrac{x^2}{1+x^2} = 1$. To see (1) that $M = 1$ is an upper bound, we have

$$g(x) = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} \le 1 - 0 = 1$$

   for all $x \in \mathbf{R}$. To see (2) that no smaller number is an upper bound, choose $s < 1$. If $s > 0$ let $x = 2(1-s)^{-1/2}$. Then

$$g(x) = 1 - \frac{1}{1+x^2} > 1 - \frac{1}{x^2} = 1 - \frac{1}{4}(1-s) = \frac{3}{4} + \frac{s}{4} > s.$$

   If $s \le 0$ let $x = 1$. In this case $g(x) = \frac{1}{2} > 0 \ge s$. In either case, there is $x \in \mathbf{R}$ such that $g(x) > s$, proving (2) holds as well.

2. *Let $\{x_n\}$ be a real sequence and L a real number. Define: $L = \lim_{n\to\infty} x_n$. Using just your definition, determine whether the limit $L = \lim_{n\to\infty} \dfrac{n-2}{3n-4}$ exists and prove your answer.*

   For the real sequence $\{x_n\}$ and real number L we say $L = \lim_{n\to\infty} x_n$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|x_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

   We claim $\lim_{n\to\infty} \dfrac{n-2}{3n-4} = \dfrac{1}{3}$. To see it, choose $\varepsilon > 0$. Let $N = \max\{4, \frac{1}{3\varepsilon}\}$. Then for every $n \in \mathbb{N}$ such that $n > N$ we have

$$\begin{aligned}|x_n - L| &= \left|\frac{n-2}{3n-4} - \frac{1}{3}\right| = \left|\frac{3(n-2)-(3n-4)}{3(3n-4)}\right| = \left|\frac{-2}{3(3n-4)}\right| = \frac{2}{3(3n-4)}\\ &\le \frac{2}{3(3n-n)} = \frac{1}{3n} < \frac{1}{3N} \le \frac{1}{3[1/(3\varepsilon)]} = \varepsilon,\end{aligned}$$

   using $3n - 4 > 3n - n > 0$ since $n > 4$ and using $N \ge \frac{1}{3\varepsilon}$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) *Let $\{x_n\}$ be a real sequence such that $x_{n+1} > x_n$ for all n. Then $\lim_{n\to\infty} x_n = \infty$.*

FALSE. The sequence $x_n = -\dfrac{1}{n}$ is strictly increasing and bounded above by 0.

(b) *Let $\{x_n\}$ be a convergent sequence such that every x_n is rational. Then the limit $\lim_{n\to\infty} x_n$ must be rational.*

FALSE. The rational sequence constructed in class and in the text from Newton's Method to find the positive root of $f(x) = x^2 - 2$, namely given recursively by $x_1 = 3$ and $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$ is a monotonically decreasing sequence that is bounded below and converges to $\sqrt{2}$, which is irrational. Another example is the sequence of rational partial sums that converge to the irrational number e:

$$y_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

A third example is gotten by taking the the truncations of the decimal expansion of an irrational number, e.g.,

$$\begin{aligned} z_1 &= 1.4 \\ z_2 &= 1.41 \\ z_3 &= 1.414 \\ z_4 &= 1.4142 \\ z_5 &= 1.41421 \\ z_6 &= 1.414213 \\ z_7 &= 1.4142135 \\ &\vdots \end{aligned}$$

(c) *There is no injective function from the real numbers to the rational numbers.*

TRUE. If there were an injective function $f : \mathbf{R} \to \mathbb{Q}$ then $\mathbf{R}$ would be dominated by $\mathbb{Q}$ ($\mathbf{R} \preccurlyeq \mathbb{Q}$) or the cardinality of $\mathbb{Q}$ is at least as large as the cardinality of $\mathbf{R}$, which is false, since $\mathbb{Q}$ is countable whereas $\mathbf{R}$ is uncountable.

4. *Let $\{a_b\}$ and $\{b_n\}$ be two real sequences that converge to real numbers a and b:*

$$a = \lim_{n\to\infty} a_n, \qquad b = \lim_{n\to\infty} b_n.$$

Using just the definition of convergence (and not the Main Limit Theorem), prove that the sequence $\{a_n|b_n|\}$ converges and

$$a|b| = \lim_{n\to\infty} a_n|b_n|.$$

The proof is like proving that the limit of a product is the product of a limit. Since $\{b_n\}$ is convergent, then it is bounded: there is an $M \in \mathbf{R}$ such that $|b_n| \le M$ for all n. Choose $\varepsilon > 0$. By the convergencce of $\{a_b\}$ and $\{b_n\}$ there are $N_1, N_2 \in \mathbf{R}$ such that

$$|a_n - a| < \frac{\varepsilon}{2M+1} \qquad \text{whenever } n > N_1 \text{ and}$$
$$|b_n - b| < \frac{\varepsilon}{2|a|+1} \qquad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\}$. For any $n \in \mathbb{N}$ such that $n > N$ we have

$$\begin{aligned}
\big|a_n|b_n| - a|b|\big| &= \big|a_n|b_n| - a|b_n| + a|b_n| - a|b|\big| \\
&\le \big|a_n|b_n| - a|b_n|\big| + \big|a|b_n| - a|b|\big| \qquad \text{Use Triangle Inequality} \\
&= \big|(a_n - a)|b_n|\big| + \big|a(|b_n| - |b|)\big| \\
&= |a_n - a|\,|b_n| + |a|\,\big||b_n| - |b|\big| \\
&\le |a_n - a|\,M + |a|\,|b_n - b| \qquad \text{Use } |b_n| \le M \text{ and Reverse Triangle Ineq.} \\
&\le \frac{\varepsilon}{2M+1}\,M + |a|\,\frac{\varepsilon}{2|a|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

5. *Let $0 < a < 1$ and define the sequence $\{x_n\}$ recursively by $x_1 = 0$ and*

$$x_{n+1} = 1 - \frac{a}{1+x_n}.$$

Prove that $\{x_n\}$ is bounded above. Prove that $\{x_n\}$ is strictly increasing. Is $\{x_n\}$ convergent? Why? If $x_n \to L$ as $n \to \infty$, what is L?

First we observe that $x_n \ge 0$ for all n. We see this by an induction argument: $x_1 = 0 \ge 0$ by prescription. Assuming $x_n \ge 0$ we get

$$x_{n+1} = 1 - \frac{a}{1+x_n} \ge 1 - \frac{a}{1+0} = 1 - a > 0.$$

Second we observe each term is bounded above by one: for every n,

$$x_{n+1} = 1 - \frac{a}{1+x_n} < 1 - 0 = 1$$

since $x_n \ge 0$ implies

$$\frac{a}{1+x_n} > 0.$$

Third we show x_n is strictly increasing by induction. For the base case,

$$x_2 = 1 - \frac{a}{1+x_1} = 1 - \frac{a}{1+0} = 1 - a > 0 = x_1.$$

For the induction case, assume $x_{n+1} - x_n > 0$ for some n. Then

$$\begin{aligned}
x_{n+2} - x_{n+1} &= \left(1 - \frac{a}{1 + x_{n+1}}\right) - \left(1 - \frac{a}{1 + x_n}\right) \\
&= \frac{a}{1 + x_n} - \frac{a}{1 + x_{n+1}} \\
&= \frac{a(1 + x_{n+1} - 1 - x_n)}{(1 + x_n)(1 + x_{n+1})} \\
&= \frac{a(x_{n+1} - x_n)}{(1 + x_n)(1 + x_{n+1})} > 0
\end{aligned}$$

by the induction hypothesis and positivity of the denominator. Thus we have shown by induction that $x_{n+1} > x_n$ for all n: $\{x_n\}$ is strictly increasing.

Thus $\{x_n\}$ is a increasing sequence which is bounded above. By the Monotone Convergence Theorem, the limit exists: $x_n \to L$ as $n \to \infty$, where L is a real number. To find L we take the recursion formula

$$x_{n+1} = 1 - \frac{a}{1 + x_n}$$

to the limit. The left side is a subsequence and the right converges by the Main Limit Theorem.

$$L = 1 - \frac{a}{1 + L}$$

Hence

$$a = (1 - L)(1 + L) = 1 - L^2$$

so, since $L \geq x_n \geq 0$,

$$L = +\sqrt{1 - a}.$$