Math 3210 § 3. **Third Midterm Exam** Name: Solutions
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1. *Let $I \subset \mathbf{R}$ be an interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is continuous on I. Using just your definition from (a) and not the combinations theorem, prove that $f(x) = \dfrac{1}{x^2}$ is continuous on $I = (0,1)$.*

   f is said to be continuous on I if for all $a \in I$ and for all $\varepsilon > 0$ there is a $\delta > 0$ such that

   $$|f(x) - f(a)| < \varepsilon \qquad \text{whenever } x \in I \text{ and } |x - a| < \delta.$$

   Choose an $a \in I$ and an $\varepsilon > 0$. Let $\delta = \min\left\{\frac{a}{2}, \frac{a^3\varepsilon}{10}\right\}$. Then for any $x \in I$ such that $|x - a| < \delta$ we have

   $$x = a + (x-a) \ge a - |x-a| > a - \delta \ge a - \frac{a}{2} = \frac{1}{2}a.$$

   and

   $$x = a + (x-a) \le x + |x-a| \le a + \delta \le a + \frac{a}{2} = \frac{3}{2}a.$$

   These inequalities imply

   $$\begin{aligned} |f(x) - f(a)| &= \left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x^2 - a^2|}{x^2a^2} = \frac{(x+a)\,|x-a|}{x^2a^2} \\ &\le \frac{(\frac{3}{2}a + a)\,|x-a|}{\frac{1}{4}a^2 \cdot a^2} = \frac{10\,|x-a|}{a^3} < \frac{10\,\delta}{a^3} \le \frac{10}{a^3}\cdot\frac{a^3\varepsilon}{10} = \varepsilon. \end{aligned}$$

   This proves $f(x)$ is continuous on I. Note that δ must not depend on x but may depend on both ε and a as it does here.

2. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is differentiable at $a \in I$. Using just your definition of derivative and properties of limits of functions, prove carefully that if $f : I \to \mathbf{R}$ is differentiable at $a \in I$ and $f(a) > 0$ then $g(x) = \sqrt{f(x)}$ is differentiable at $a \in I$ and find $g'(a)$.*

   $f : I \to \mathbf{R}$ is said to be differentiable at $a \in I$ if the limit

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists and equals some $f'(a) \in \mathbf{R}$.

   We assume that $f(x)$ is differentiable at $a \in I$. That means

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists where $f'(a) \in \mathbf{R}$. Now we compute the limit of the difference quotient for $g(x)$ at $z \in I$.

   $$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{f(x)} - \sqrt{f(a)}}{x - a}$$

Since $f(x)$ is differentiable at a it is continuous at a. Since $f(a) > 0$, for every x close enough to a we also have $f(x) > 0$. Thus we may rationalize the numerator since we do not divide by zero

$$\begin{aligned}\lim_{x\to a}\frac{\sqrt{f(x)}-\sqrt{f(a)}}{x-a} &= \lim_{x\to a}\frac{\left(\sqrt{f(x)}-\sqrt{f(a)}\right)\left(\sqrt{f(x)}+\sqrt{f(a)}\right)}{(x-a)\left(\sqrt{f(x)}+\sqrt{f(a)}\right)}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{(x-a)}\cdot\frac{1}{\left(\sqrt{f(x)}+\sqrt{f(a)}\right)}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{(x-a)}\cdot\lim_{x\to a}\frac{1}{\left(\sqrt{f(x)}+\sqrt{f(a)}\right)}\\ &= f'(a)\cdot\frac{1}{\left(\sqrt{f(a)}+\sqrt{f(a)}\right)} = \frac{f'(a)}{2\sqrt{f(a)}},\end{aligned}$$

where we have used the fact that the limit of a product is the product of limits and the limit of a quotient is the quotient of limits provided that the denominator is nonzero. Hence $g(x)$ is differentiable at $a \in I$ and the derivative is as expected from the chain rule.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *If f is differentiable on the bounded open interval (a,b) and satisfies $|f'(x)| \le 100$ for all $x \in (a,b)$ then f is uniformly continuous.*

   TRUE. We can show using the mean Value theorem that f is 100-Lipschitz, hence uniformly continuous. For any $x, y \in I$, say $x < y$, we have f continuous on $[x,y]$ and differentiable on (x,y) because it is differentiable on $[x,y] \subset (a,b)$. Hence, by the Mean Value Theorem, there is a $c \in (x,y)$ such that

   $$f(x) - x(y) = f'(c)\,(x-y).$$

   Taking absolute values

   $$|f(x) - x(y)| = |f'(c)|\,|x-y| \le 100\,|x-y|.$$

   (b) *There is a positive real number c such that $c = \tan c$.*

   TRUE. Let $g(x) = \tan x - x$ which is continuous on the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Since tangent tends to $-\infty$ as $x \to \frac{\pi}{2}+$ and to $+\infty$ as $x \to \frac{3\pi}{2}-$, there are $\frac{\pi}{2} < a < b < \frac{3\pi}{2}$ such that $g(a) < 0$ and $g(b) > 0$. Thus by the Intermediate Value Theorem, there is $c \in (a,b)$ such that $g(c) = 0$.

   (c) *Let $f, g : (1,2) \to \mathbf{R}$ be differentiable functions such that $g(x) \ne 0$ and $g'(x) \ne 0$ for all $x \in (1,2)$ and such that $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} g(x)$ exist and are equal.*

   If $L = \lim_{x\to 1+}\frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to 1+}\frac{f(x)}{g(x)}$ exists and equals L.

   FALSE. L'Hopital's Rule does not apply since we have not specified that the limit is is of "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" type. Thus if we pick two functions $f(x) = 2x$ and $g(x) = 1 + x$ which are differentiable on $(1,2)$ and $g(x)$ and $g'(x)$ are nonzero on $(0,1)$ for which the limits both exist

   $$L = \lim_{x\to 1+}\frac{f'(x)}{g'(x)} = \lim_{x\to 1+}\frac{2}{1} = 2$$

   but

   $$\lim_{x\to 1+}\frac{f(x)}{g(x)} = \lim_{x\to 1+}\frac{2x}{1+x} = 1.$$

4. *Let $I \subset \mathbf{R}$ be a closed and bounded interval and $f, f_n : I \to \mathbf{R}$ be functions. State the definitions:*

(a) $\{f_n(x)\}$ converges pointwise to a function f on I as $n \to \infty$.

(a) $\{f_n(x)\}$ *converges uniformly* to a function f on I as $n \to \infty$.

Determine whether the functions $f_n(x) = x^{1/n}$ converge pointwise or uniformly to a function $f(x)$ on the interval $I = [0, 1]$ and prove your result.

$\{f_n(x)\}$ converges pointwise to a function f on I as $n \to \infty$ if for all $a \in I$, $\lim_{n\to\infty} f_n(a) = f(a)$. In other words, for all $a \in I$ and all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|f_n(a) - f(a)| < \varepsilon \qquad \text{whenever } n > N.$$

$\{f_n(x)\}$ converges uniformly to a function f on I as $n \to \infty$ if for all $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|f_n(a) - f(a)| < \varepsilon \qquad \text{whenever } a \in I \text{ and } n > N.$$

Note that for $f_n(x) = x^{1/n}$ we have $f_n(0) = 0$ for all n which tends to $f(0) = 0$. If $x > 0$ then $f_n(x)$ tends to one as $n \to \infty$. Thus the functions converge to the pointwise limit

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

The convergence is not uniform. If we take the sequence $x_n = 2^{-n} \in I$ then $f_n(x_n) = \frac{1}{2}$ for all n. Thus

$$|f_n(x_n) - f(x_n)| = \left| \frac{1}{2} - 1 \right| = \frac{1}{2}$$

which does not tend to zero, which it must do if the convergence were uniform. Or one could observe that f is a discontinuous limit of continuous functions, which is impossible if the convergence were uniform. Another proof may involve showing that the negation of uniform convergence holds.

Note that the inverse functions are $f_n^{-1}(y) = y^n$ for $y \in I$, making f_n easy to graph. This is the standard example of a pointwise convergent sequence that is not uniformly convergent.

5. *For the bounded sequence $\{b_n\}$, let*

$$S_n = b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} = \sum_{k=0}^{n} \frac{b_k}{2^k}.$$

Define: $\{S_n\}$ is a Cauchy Sequence. *Show that there is an $L \in \mathbf{R}$ such that $S_n \to L$ as $n \to \infty$.*

$\{S_n\}$ is said to be a Cauchy Sequence if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|S_p - S_q| < \varepsilon \qquad \text{whenever } p > N \text{ and } q > N.$$

We show that the given sequence $\{S_n\}$ is a Cauchy Sequence, hence $S_n \to L$ converges to an $L \in \mathbf{R}$ as $n \to \infty$.

Because $\{b_n\}$ is bounded, there is an $M \in \mathbf{R}$ such that $|b_n| \le M$ for all n. Choose $\varepsilon > 0$. Let $N \in \mathbf{R}$ satisfy $M2^{-N} < \varepsilon$. Then for any $p, q \in \mathbb{N}$ such that $p > N$ and $q > N$ we may suppose that $p > q$. If $p = q$ then $|S_p - S_q| = 0 < \varepsilon$. If $p < q$ then we may swap the roles of p and q in the proof. Then by the triangle inequality,

$$|S_p - S_q| = \left|\sum_{k=0}^{p} \frac{b_k}{2^k} - \sum_{k=0}^{q} \frac{b_k}{2^k}\right| = \left|\sum_{k=q+1}^{p} \frac{b_k}{2^k}\right| \leq \sum_{k=q+1}^{p} \frac{|b_k|}{2^k} \leq M \sum_{k=q+1}^{p} \frac{1}{2^k}$$

$$= \frac{M}{2^{q+1}} \sum_{k=0}^{p-q-1} \frac{1}{2^k} = \frac{M}{2^{q+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{p-q}}{1 - \frac{1}{2}} \leq \frac{M}{2^q} \leq \frac{M}{2^N} < \varepsilon.$$

This shows that $\{S_n\}$ is a Cauchy sequence, thus convergent.