Math 3210 § 2. **First Midterm Exam** Name: Solutions

Treibergs February 2, 2022.

1. *Prove that for every natural number* n

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}. \tag{1}$$

Proof by induction.

In the base case $n = 1$ the left side and right sides are equal.

$$\text{LHS} = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}; \qquad \text{RHS} = \frac{1}{1+1} = \frac{1}{2}.$$

In the induction case, we assume that for some $n \in \mathbb{N}$ that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Then for $n+1$, using the induction hypothesis

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \left(\sum_{k=1}^{n} \frac{1}{k(k+1)} \right) + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{(n+1)+1},
\end{aligned}$$

which is the statement for $n+1$. We conclude by induction that (1) holds for all $n \in \mathbb{N}$. $\square$

2. *Recall the axioms of a field* $(\mathcal{F}, +, \times)$. *For any* $x, y, z \in \mathcal{F}$,

[A1.]	(Commutativity of Addition)	$x + y = y + x$.
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z$.
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\ (\forall\, t \in \mathcal{F})\ 0 + t = t$.
[A4.]	(Additive Inverse)	$(\exists\, {-x} \in \mathcal{F})\ x + (-x) = 0$.
[M1.]	(Commutativity of Multiplication)	$xy = yx$.
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z$.
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})$ $1 \neq 0$ and $(\forall\, t \in \mathcal{F})$ $1t = t$.
[M4.]	(Multiplicative Inverse)	If $x \neq 0$ then $(\exists\, x^{-1} \in \mathcal{F})$ $(x^{-1})x = 1$.
[D.]	(Distributivity)	$x(y + z) = xy + xz$.

Using only the field axioms, show that for any $a \in \mathcal{F}$ *the additive inverse* $-a$ *is unique. Justify every step of your argument using just the axioms listed here. Do not quote any formulas from the text.*

We suppose that there are two additive inverses for some $a \in \mathcal{F}$, call them $-a$ and w and argue they must be the same number. Both satisfy axiom (A3), namely

$$a + (-a) = 0 \qquad \text{and} \qquad a + w = 0.$$

Both expressions equal zero, thus they equal each other.

$a + (-a) = a + w$	Both expressions are equal.
$(-a) + a = w + a$	Commutativity of addition. (A2)
$((-a) + a) + (-a) = (w + a) + (-a)$	Additive inverse: there is a $(-a)$ such that $a + (-a) = 0$. Post-add it to both sides. (A4)
$(-a) + (a + (-a)) = w + (a + (-a))$	Associativity of addition. (A2)
$(-a) + 0 = w + 0$	Additive inverse. (A4)
$0 + (-a) = 0 + w$	Commutivity of addition. (A1)
$-a = w$	Additive identity. (A3)

We conclude that the two additive inverses are equal. Thus the additive invere is unique.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement. *Let* $f : A \to B$ *and* $E, G \subset A$. *Then* $f(E) \backslash f(G) = f(E \backslash G)$.
   False. The equality holds if and only if f is one-to-one. A counterexample is given by any not one-to-one function, such as $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. Let $E = [-3, -2) \cup (2, 3]$ and $G = (2, 3]$. Then $f(E) = f(G) = (4, 9]$ so $f(E) \backslash f(G) = \emptyset$ but $E \backslash G = [-3, -2)$ and $f(E \backslash G) = (4, 9]$.

   (b) Statement. *If* $f : \mathbf{R} \to \mathbf{R}$ *is function of the real numbers such that* $x \neq y$ *implies* $f(x) \neq f(y)$ *for all* $x, y \in \mathbf{R}$, *then* f *is* onto.
   False. This is the definition of one-to-one, not onto. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(z) = \dfrac{x}{\sqrt{1 + x^2}}$ which is strictly increasing. Then $x \neq y$, say $x < y$, implies $f(x) < f(y)$ so the condition holds but $f(\mathbf{R}) = (-1, 1) \neq \mathbf{R}$ so f is not onto.

(c) STATEMENT. *Let x be a real number such that $x > 0$. Then there is a rational number $r \in \mathbb{Q}$ such that $0 < r < x$.*

TRUE. This follows from the Archimedean property of $\mathbf{R}$. Given $x > 0$ there is $n \in \mathbb{N}$ so that $r = \dfrac{1}{n} < x$. But r is rational.

4. *Recall that the rational numbers are defined to be the set of equivalence classes $\mathbb{Q} = S/\sim$ where $S = \left\{\dfrac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\right\}$ is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction: $\dfrac{a}{b} \sim \dfrac{c}{d}$ iff $ad = bc$. We denote the equivalence class, the "fraction" by $\left[\dfrac{a}{b}\right]$ to distinguish it from a symbol from S. Addition and multiplication of rationals, for example, is defined on equivalence classes by*

$$\left[\frac{m}{n}\right] + \left[\frac{r}{t}\right] = \left[\frac{mt + nr}{nt}\right], \qquad \left[\frac{m}{n}\right] \cdot \left[\frac{r}{t}\right] = \left[\frac{mr}{nt}\right].$$

How is order $\left[\dfrac{p}{q}\right] \geq 0$ defined in the rationals? Explain what it means that $\left[\dfrac{p}{q}\right] \geq 0$ is well defined. Is $\sqrt{5}$, the square root of 5 rational? Explain why or why not.

We say $\left[\dfrac{p}{q}\right] \geq 0$ whenever the symbol is equivalent to another symbol $\dfrac{p}{q} \sim \dfrac{p'}{q'}$ where $p' > 0$ and $p' \geq 0$. Equivalently $pq \geq 0$. This notion is well defined if whenever there is a symbol in the same equivalence class $\dfrac{p}{q} \sim \dfrac{p''}{q''}$, then $\left[\dfrac{p}{q}\right] \geq 0$ if and only if $\left[\dfrac{p''}{q''}\right] \geq 0$. In fact this follows from the fact that $\dfrac{p}{q} \sim \dfrac{p'}{q'}$ and $\dfrac{p}{q} \sim \dfrac{p''}{q''}$ then $\dfrac{p''}{q''} \sim \dfrac{p'}{q'}$.

NO. To see $\sqrt{5}$ is not rational, we may argue in two ways.

The first way is to use the theorem in the text that says if $x^2 = k$ where k is an integer and x is rational, then x is an integer. Thus if x were rational, then the solution of $x^2 = 5$ is an integr j. But the squares of the small integers $j = 0, \pm 1, \pm 2, \pm 3$ are $0, 1, 4, 9$, resp. And if $|j| > 3$ then $j^2 > 9$. It follows no square of $x = j$ is 5, thus $\sqrt{5}$ cannot be rational.

The second way is the usual contradiction proof of irrationality. Assuming $x = \sqrt{5} = \dfrac{p}{q}$ is rational, after cancelling factors we may assume that p and q have no common factors. Now $x^2 = 5$ implies $p^2 = 5q^2$. But this says 5 divides p^2. Since 5 is prime, 5 divides p and we may write $p = 5k$ for some integer k. Hence $25k^2 = 5q^2$ or $5k^2 = q^2$. Again this says 5 divides q^2. But since 5 is prime, 5 divides q. So 5 is a factor of both p and q, contrary to our assumption about p and q. We conclude that $\sqrt{5}$ could not have been rational.

5. *Let* $E \subset \mathbb{R}$ *be a set of real numbers. Find the least upper bound of* E *and prove your assertion.*

$$E = \left\{ \frac{2^n - 1}{2^n} : n \in \mathbb{N} \right\}$$

$L = \operatorname{lub} E = 1$. To see this we have to show that $L = 1$ is an upper bound and that no smaller number is an upper bound. For all n we have

$$\frac{2^n - 1}{2^n} \le \frac{2^n}{2^n} = 1,$$

so $L = 1$ is an upper bound for E.

Suppose that $b < 1$ to show that there is $x \in E$ such that $b < x$, hence b is not an upper bound. Since $1 - b > 0$, by the Archimidean Property, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < 1 - b.$$

On the other hand, we know that for all $n \in \mathbb{N}$ we have $n < 2^n$. It follows for $n = n_0$ that

$$x = \frac{2^{n_0} - 1}{2^{n_0}} = 1 - \frac{1}{2^{n_0}} > 1 - \frac{1}{n_0} > 1 - (1 - b) = b.$$

Thus we have $x \in E$ such that $b < x$. Thus no number $b < 1$ is an upper bound for E so $L = 1$ is the least upper bound of E..