Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs March 2, 2022

1. *Let $f : I \to \mathbf{R}$ be a real valued function defined on the interval $I = (0, 1)$.*
   Define: $M = \inf_{\mathbf{I}} f$. Let $g(x) = \dfrac{1}{x}$. Find $M = \inf_I g$ and prove your result.

   If f is not bounded below, then we define $\inf_{\mathbf{I}} f = -\infty$. If f is bounded below, then the real number $M = \inf_{\mathbf{I}} f$ if (1) M is a lower bound, that is $M \le f(x)$ for all $x \in I$ and (2) M is the greatest of all lower bounds. In other words, no larger number is a lower bound, that is, for every $b > M$ there is $z \in I$ so that $f(z) < b$.

   The function $g(x) = \dfrac{1}{x}$ is decreasing on the interval I thus we claim $\inf_{\mathbf{I}} g = 1$. To see that 1 is a lower bound, we observe that whenever $x \in I$, that is $0 < x < 1$ then $1 < x^{-1} = f(x)$, so $M = 1$ is a lower bound. To see that no greater number is a lower bound, choose $b > 1$. Then $0 < \dfrac{1}{b} < 1$ and $f\left(\dfrac{1}{b}\right) = b$. Thus if we choose any number $\dfrac{1}{b} < z < 1$, such as the average $z = \dfrac{1}{2}\left(\dfrac{1}{b} + 1\right)$, then $z \in I$ and since f is decreasing, $b = f\left(\dfrac{1}{b}\right) > f(z)$. Thus there is $z \in I$ so that $f(z) < b$.

2. *Let $\{x_n\}$ be a real sequence and a a real number. Define: $a = \lim\limits_{n\to\infty} x_n$. Find L and, using just your definition, prove that L is the limit. $L = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+4n^2}}{1+n}$.*

   $a \in \mathbf{R}$ is the limit, $a = \lim\limits_{n\to\infty} x_n$ means that for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that

   $$|x_n - a| < \epsilon \qquad \text{whenever } n > N.$$

   By the Main Limit Theorem, we see that

   $$L = \lim_{n\to\infty} \frac{\sqrt{1+4n^2}}{1+n} = \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^2} + 4}}{\frac{1}{n} + 1} = \frac{\sqrt{0+4}}{0+1} = 2.$$

   To prove that L is the limit, we choose $\epsilon > 0$. Let $N = \dfrac{4}{\epsilon}$. Then for any $n \in \mathbb{N}$ such that $n > N$ we have by rationalizing the numerator

   $$\begin{aligned}
   |x_n - L| &= \left| \frac{\sqrt{1+4n^2}}{1+n} - 2 \right| = \left| \frac{\sqrt{1+4n^2} - 2(n+1)}{1+n} \right| \\
   &= \left| \frac{[\sqrt{1+4n^2} - 2(n+1)]}{1+n} \cdot \frac{[\sqrt{1+4n^2} + 2(n+1)]}{[\sqrt{1+4n^2} + 2(n+1)]} \right| \\
   &= \frac{\left|[1+4n^2] - 4(n+1)^2\right|}{[1+n]\,[\sqrt{1+4n^2} + 2(n+1)]} \\
   &\le \frac{|-8n-3|}{2(1+n)^2} \le \frac{8n+8}{2(1+n)^2} = \frac{4}{1+n} < \frac{4}{1+N} < \frac{4}{N} = \epsilon.
   \end{aligned}$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENBT: *Let $\{x_n\}$ and $\{y_n\}$ be a real sequences such that $y_n - x_n = \dfrac{1}{2^n}$ for all n . Then $\emptyset \ne \displaystyle\bigcap_{n=1}^{\infty} [x_n, y_n]$.*

   FALSE. The statement would be true if assumed that the intervals were nested, $[x_1, y_1] \supset [x_2, y_2] \supset [x_3, y_3] \supset \cdots$ by the Nested Intervals Theorem. However that was not given, so we construct some disjoint intervals of the right lengths whose intersection is empty, *e.g.,* $[x_1, y_1] = [0, .5]$, $[x_2, y_2] = [1, 1.25]$ and $[x_n, y_n] = [2, 2 + \frac{1}{2^n}]$ for $n \ge 3$. Then $\emptyset = \bigcap_{n=1}^{\infty} [x_n, y_n]$.

   (b) STATEMENT: *If the sequence $\{x_n\}$ is not convergent, then none of its subsequences is convergent.*

   FALSE. The sequence $x_n = (-1)^n$ is not convergent but has the convergent subsequence $x_{2n} = (-1)^{2n} = 1 \to 1$ as $n \to \infty$.

   (c) STATEMENT: *Suppose the sequence $\{a_n\}$ satisfies $a_n \ne 0$ for all n and converges to $a \in \mathbf{R}$. Then the sequence $\{1/a_n\}$ converges.*

   FALSE. The statement would be true by the Main Limit Theorem if the additional assumption that $a \ne 0$ held. But this was not given. If we take $a_n = \dfrac{(-1)^n}{n} \to 0$ as $n \to 0$ we would have $\dfrac{1}{a_n} = (-1)^n n$ which does not converge.

4. *Let $\{a_n\}$ be a real sequences that converges $a = \lim_{n\to\infty} a_n$ where $a < 10$. Using just the definition of convergence, prove there is an $N \in \mathbf{R}$ so that $a_n < 10$ whenever $n > N$.*

   Let $\epsilon = 10 - a > 0$. Since we assume $a_n \to a$ as $n \to \infty$, there is an $N \in \mathbf{R}$ such that

   $$|a_n - a| < \epsilon \qquad \text{whenever } n > N.$$

   For this N, if $n \in \mathbb{N}$ is any number such that $n > N$, then

   $$a_n = a + (a_n - a) \le a + |a_n - a| < a + \epsilon = a + (10 - a) = 10.$$

5. *Define: $\{x_n\}$ is a* Cauchy Sequence. *Let $x_n = \displaystyle\sum_{k=1}^{n} \frac{\sin(k)}{k^k}$. Prove that $\{x_n\}$ is convergent.*

   A sequence $\{x_n\}$ is called a Cauchy Sequence if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ so that

   $$|x_i - x_j| < \epsilon \qquad \text{whenever } i > N \text{ and } j > N.$$

   We show that the given sequence satisfies the Cauchy Criterion, hence is convergent. Choose $\epsilon > 0$. Let $N \in \mathbb{N}$ be large enough so that $2^{-N} < \epsilon$. For any $i, j \in \mathbb{N}$ such that $i > N$ and $j > N$ we may assume that $i > j$. If $i = j$ then $|a_i - a_j| = 0 < \epsilon$. If $i < j$ then interchange a_i and a_j. We have for such $i > j$, using $|\sin k| \le 1$ and $k^k \ge 2^k$ whenever $k \ge 2$,

   $$|a_i - a_j| = \left| \sum_{k=1}^{i} \frac{\sin(k)}{k^k} - \sum_{k=1}^{j} \frac{\sin(k)}{k^k} \right| = \left| \sum_{k=j+1}^{i} \frac{\sin(k)}{k^k} \right| \le \sum_{k=j+1}^{i} \frac{|\sin(k)|}{k^k}$$

   $$\le \sum_{k=j+1}^{i} \frac{1}{2^k} = \frac{1}{2^{j+1}} \sum_{\ell=0}^{i-j-1} \frac{1}{2^\ell} = \frac{1}{2^{j+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{i-j}}{1 - \frac{1}{2}} \le \frac{1}{2^j} < \frac{1}{2^N} < \epsilon.$$