Math 3210 § 2. **Third Midterm Exam** Name: Solutions
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1. *Let $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$. State the definition: $f(x)$ is* continuous at a. Using just your definition and not the combinations theorem, prove that $f(x) = (x+1)^2$ is continuous at $a \in \mathbf{R}$.

   $f : \mathbf{R} \to \mathbf{R}$ is said to be continuous at $a \in \mathbf{R}$ is for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } |x - a| < \delta.$$

   Choose $\epsilon > 0$. Let $\delta = \min\left\{\dfrac{\epsilon}{3 + 2|a|}, 1\right\}$. Then for any $x \in \mathbf{R}$ such that $|x - a| < \delta$ we have since $\delta \le 1$,

$$|x + a + 2| = |(x - a) + 2a + 2| \le |x - a| + 2|a| + 2 < \delta + 2|a| + 2 \le 3 + 2|a|.$$

   Using this inequality,

$$\begin{aligned} |f(x) - f(a)| &= |(x+1)^2 - (a+1)^2| \\ &= |(x^2 + 2x + 1) - (a^2 + 2a + 1)| \\ &= |x^2 - a^2 + 2(x - a)| \\ &= |(x + a + 2)(x - a)| \\ &= |x + a + 2|\,|x - a| \\ &< (3 + 2|a|)\delta \\ &\le (3 + 2|a|) \cdot \frac{\epsilon}{3 + 2|a|} = \epsilon. \end{aligned}$$

2. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is* differentiable *at $a \in I$. Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies $|f(x)| \le |x|^{3/2}$. Using just your definition of derivative and properties of limits of functions, prove carefully that f is differentiable at $a = 0$, and find $f'(0)$.*

   $f : I \to \mathbf{R}$ is said to be *differentiable at $a \in I$* if the following limit exists and is finite:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

   Since $|f(x)| \le |x|^{3/2}$, we have $|f(0)| \le |0|^{3/2} = 0$ so $f(0) = 0$. We show that the derivative exists and is zero $f'(0) = 0$. Indeed, for $x \ne 0$,

$$-|x|^{1/2} = -\frac{|x|^{3/2}}{|x|} \le \frac{f(x) - f(a)}{x - a} = \frac{f(x) - 0}{x - 0} \le \frac{|x|^{3/2}}{|x|} = |x|^{1/2}$$

   Letting $x \to 0$, square roots at the ends tend to zero so by the Squeeze Theorem, the expression in the middle tends to $f'(0) = 0$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMNET: *If f is differentiable and $f' > 0$ everywhere in $\mathbf{R}$, then f has an inverse function which is differentiable and strictly increasing on $f(\mathbf{R})$.*

   TRUE. The fact that f is differentiable at all x implies f is continuous at all x. The assumption that $f'(x) > 0$ for all x says f is strictly increasing on $\mathbf{R}$ so its image $J = f(\mathbf{R})$ is an interval and f has a a contnuous inverse function $f^{-1} : J \to \mathbf{R}$. At all corresponding points $f(a) = b$, because f is differentiable and $f'(a) \neq 0$, the derivative of the inverse function exists and satisfies $\dfrac{d}{dy} f^{-1}(b) = \dfrac{1}{f'(a)} > 0$. Hence f^{-1} is differentiable at all $b \in J$ and is strictly increasing on J.

   (b) STATEMENT: *If $f(x) = \dfrac{x^5 - 2x^2 + 3}{x^4 - 2x^2 + 3}$, then there is a real number c such that $f(c) = 0$.*

   TRUE. Note that the denominator $x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2 > 0$ for all x so that the rational function $g(x)$ is continuous on $\mathbf{R}$. Since $f(0) = 1 > 0$ and $f(-2) = -\dfrac{37}{11} < 0$ the value 0 is between these two, so by the Intermediate Value Theorem, there is $c \in (-2, 0)$ such that $f(c) = 0$.

   (c) STATEMENT: *Suppose the function $f(x)$ is a bounded and continuous for $x \in (0, 1)$. Suppose $\{x_n\}$ be a sequence in $(0, 1)$ such that $x_n \to 1$ as $n \to \infty$. Then the limit $L = \lim_{n\to\infty} f(x_n)$ exists and is finite.*

   FALSE. It would be true if f were uniformly continuous. For a continuous counterexample, consider the function $f(x) = \cos\left(\dfrac{1}{1-x}\right)$ which is continuous and bounded, $|f(x)| \leq 1$, on $(0, 1)$. For the sequence $x_n = 1 - \dfrac{1}{\pi n} \to 1$ as $n \to \infty$, we have $f(x_n) = (-1)^n$, which does not converge as $n \to \infty$.

4. *Let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions. State the definitions:*

   (a) $\{f_n(x)\}$ converges pointwise *to a function f on $\mathbf{R}$ as $n \to \infty$.*

   (b) $\{f_n(x)\}$ converges uniformly *to a function f on $\mathbf{R}$ as $n \to \infty$.*

   Determine whether the functions $f_n(x) = \dfrac{nx}{n^2 + x^2}$ converge pointwise, converge uniformly, or do not converge to a function $f(x)$ on the $\mathbf{R}$ and prove your result.

   The sequence of functions $f_n \to f$ is said to *converge pointwise* on $\mathbf{R}$ if for every $a \in \mathbf{R}$ and every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|f_n(a) - f(a)| < \epsilon \qquad \text{whenever } n > N.$$

   The sequence of functions $f_n \to f$ is said to *converge uniformly* on $\mathbf{R}$ if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|f_n(a) - f(a)| < \epsilon \qquad \text{whenever } n > N \text{ and } a \in \mathbf{R}.$$

   The sequence $f_n(x) = \dfrac{nx}{n^2 + x^2}$ converges pointwise but does not converge uniformly to $f(x) = 0$ on $\mathbf{R}$. To see the pointwise convergence, for any $a \in \mathbf{R}$ we have

$$\lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \frac{na}{n^2 + a^2} = \lim_{n\to\infty} \frac{\frac{a}{n}}{1 + \frac{a^2}{n^2}} = \frac{0}{1+0} = 0.$$

The easiest way to see that that the convergence is not uniform, we can use the theorem that if $f_n \to 0$ uniformly on $\mathbf{R}$, then for any sequence $\{x_n\} \subset \mathbf{R}$ we have $\lim_{n\to\infty} f_n(x_n) = 0$. But this fails by choosing the sequence $x_n = n$ we find for all $n \in \mathbb{N}$,

$$f_n(x_n) = \frac{nx_n}{n^2 + x_n^2} = \frac{n^2}{n^2 + n^2} = \frac{1}{2} \not\to 0$$

as $n \to \infty$.

Another way is to prove the negation of the definition of uniform convergence $f_n \to f$, namely, to show that there is an $\epsilon_0 > 0$ such that for every $N \in \mathbf{R}$ there is an $n \in \mathbb{N}$ such that $n > N$ and an $a \in \mathbf{R}$ such that $|f_n(a) - f(a)| \ge \epsilon_0$. For the particular sequence $f_n(x) = \dfrac{nx}{n^2 + x^2}$ and $f(x) = 0$, we put $\epsilon_0 = \frac{1}{2}$. For any $N \in \mathbf{R}$, by the Archimedean Axiom, there is $n \in \mathbb{N}$ such that $n > N$ and for $a = n$ we have

$$|f_n(a) - f(a)| = \left| \frac{na}{n^2 + a^2} - 0 \right| = \frac{n^2}{n^2 + n^2} = \frac{1}{2} \ge \epsilon_0.$$

5. *For the bounded sequence $\{b_n\}$, let*

$$S_n = b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} = \sum_{k=0}^{n} \frac{b_k}{2^k}.$$

Define: $\{S_n\}$ is a Cauchy Sequence. Show that there is an $L \in \mathbf{R}$ such that $S_n \to L$ as $n \to \infty$.

The sequence $\{S_n\}$ is said to be a *Cauchy Sequence* if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|S_k - S_\ell| < \epsilon \qquad \text{whenever } k > N \text{ and } \ell > N.$$

$\{b_n\}$ bounded means that there is $B \in \mathbf{R}$ such that $|b_n| \le B$ for all n. To show that $\{S_n\}$ is a Cauchy Sequence, choose *psilon* > 0 and let $N \in \mathbf{R}$ be so large that $\dfrac{B}{2^N} < \epsilon$. Now for any $m, \ell \in \mathbb{N}$ we may suppose $m > \ell$. If instead $m = \ell$ then $|S_m - S_\ell| = 0 < \epsilon$ or if $m < \ell$ we may swap the roles of m and ℓ since $|S_m - S_\ell| = |S_\ell - S_m|$. For any $m > \ell > N$ we have by the triangle inequality and the formula for a geometric sum,

$$\begin{aligned} |S_m - S_\ell| &= \left| \sum_{k=0}^{m} \frac{b_k}{2^k} - \sum_{k=0}^{\ell} \frac{b_k}{2^k} \right| = \left| \sum_{k=\ell+1}^{m} \frac{b_k}{2^k} \right| \le \sum_{k=\ell+1}^{m} \frac{|b_k|}{2^k} \le \sum_{k=\ell+1}^{m} \frac{B}{2^k} \\ &= \frac{B}{2^{\ell+1}} \cdot \sum_{k=0}^{m-\ell-1} \frac{1}{2^k} = \frac{B}{2^{\ell+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{m-\ell}}{1 - \frac{1}{2}} < \frac{B}{2^\ell} < \frac{B}{2^N} < \epsilon. \end{aligned}$$