Math 3210 § 2. **Final Exam** Name:______________________

Treibergs April 28, 2022

This is a closed book exam except that you are allowed four "cheat sheets," four 8.5"×11" pages with notes on both sides. Other notes, books, calculators, tablets, laptops, phones and text messaging devices are prohibited. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.	/21
2.	/22
3.	/21
4.	/22
5.	/21
6.	/21
7.	/21
8.	/22
9.	/22
Total	/150

1. (a) [10] Prove that $G(x) = \displaystyle\int_0^{2x} \sqrt{1+t^3}\,dt$ is continuous at $a = 2$.

   (b) [11] Determine whether the improper integral exists.

$$\int_0^{\infty} \frac{dx}{\sqrt{x + x^2 + x^3}}$$

2. (a) [3] Let $f : (1, \infty) \to \mathbf{R}$. Define the *supremum of* f, $\displaystyle\sup_{x \in (1,\infty)} f(x)$.

(b) [19] Find $\displaystyle\sup_{x \in (1,\infty)} \frac{x}{x+1}$ and prove your result.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] $|e^x - e^y| \le 3|x-y|$ for all $x, y \in [0,1]$. TRUE: ◯ FALSE: ◯

   (b) [7] If $f : (0,1) \to \mathbb{R}$ is continuous then f assumes its maximum on $(0,1)$. TRUE: ◯ FALSE: ◯

   (c) [7] Let $f : (0,\infty) \to \mathbf{R}$ be differentiable. If $f'(x)$ is not bounded on $(0,\infty)$ then f is not uniformly continuous on $(0,\infty)$. TRUE: ◯ FALSE: ◯

4. (a) [3] Define what it means for $f : \mathbf{R} \to \mathbf{R}$ to be *differentiable* at the point $c \in \mathbf{R}$.

(b) [19] Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $c \in \mathbf{R}$ and that $f'(c) > 0$. Show that there is $\delta > 0$ so that $f(x) > f(c)$ whenever $c < x < c + \delta$.

5. For $n \in \mathbf{N}$, let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions.

   (a) [3] Define: the sequence $\{a_n\}$ is a *Cauchy Sequence.*

   (b) [18] Let $\{a_k\}$ be a real sequence such that the series $\sum_{k=1}^{\infty} |a_k|$ converges. Show that $\sum_{k=1}^{\infty} a_k$ also converges.

6. (a) [3] Let $S, f_k : [0,1] \to \mathbf{R}$. Define: $S(x) = \sum_{k=1}^{\infty} f_k(x)$ *converges uniformly* on $[0,1]$.

(b) [18] Suppose that $f_k : [0,1] \to \mathbf{R}$ are continuous functions such that $S(x) = \sum_{k=1}^{\infty} f_k(x)$ converges uniformly. Prove that $S(x)$ is a continuous function,

7. Determine whether the following series converge.

(a) [7] $S \sim \sum_{k=1}^{\infty} \frac{k^4}{(1+k^2)^3}$ | CONVERGES: ◯ | DOES NOT CONVERGE: ◯

(b) [7] $T \sim \sum_{k=1}^{\infty} \frac{e^{k^2}}{k!}$ | CONVERGES: ◯ | DOES NOT CONVERGE: ◯

(c) [7] $U \sim \sum_{k=1}^{\infty} \frac{4+\sin k}{(3+\cos k)^k}$ | CONVERGES: ◯ | DOES NOT CONVERGE: ◯

8. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Complete the statement of the theorem. [Of several possible answers, select the one you prefer for part (b).]

   Theorem. *The bounded function f is integrable on $[a, b]$ if and only if*

   (b) [19] Using only the theorem in (a), show that if $f(x)$ is integrable on $[0, 3]$ then the restriction of f to $[0, 2]$ is integrable on $[0, 2]$.

9. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Define what it means for f to be *integrable* on $[a, b]$ and what the *Riemann integral* of f on $[a, b]$ is.

   (b) [19] Find the upper integral $\overline{\int_0^2} f(x)\,dx$ and the lower integral $\underline{\int_0^2} f(x)\,dx$ where

$$f(x) = \begin{cases} 1, & \text{if } x \le \sqrt{2}; \\ \\ 2, & \text{if } x > \sqrt{2}. \end{cases}$$

   What does this say about f?