Math 3210 § 1. **Third Midterm Exam** Name: Solutions
Treibergs April 5, 2023

1. *Let $\mathcal{D} \subset \mathbf{R}$, $a \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbf{R}$. State the definition: $f(x)$ is continuous at a in $\mathcal{D}$. Using just your definition and not the combinations theorem, prove that $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous at $a \in (0, \infty)$.*

   DEFINITION. f is *continuous* at a in $\mathcal{D}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever } x \in \mathcal{D} \text{ and } |x - a| < \delta.$$

   We show that $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous at any given $a \in (0, \infty)$. Choose $\varepsilon > 0$. Let $\delta = \min\left\{ \dfrac{a}{2}, \dfrac{a^{\frac{3}{2}}\varepsilon}{\sqrt{2}} \right\}$. Suppose that $x \in (0, \infty)$ such that $|x - a| < \delta$. Because $\delta \le \dfrac{a}{2}$ it follows that

$$x = a + (x - a) \ge a - |x - a| > a - \delta \ge a - \frac{a}{2} = \frac{a}{2}. \tag{1}$$

   Hence using (1) and $\delta \le \dfrac{a^{\frac{3}{2}}\varepsilon}{\sqrt{2}}$,

$$\begin{aligned} |f(x) - f(a)| &= \left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| = \left| \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}} \right| = \left| \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}} \cdot \frac{\sqrt{a} + \sqrt{x}}{\sqrt{x} + \sqrt{a}} \right| \\ &= \frac{|a - x|}{\sqrt{x}\sqrt{a}(\sqrt{x} + \sqrt{a})} \le \frac{|a - x|}{(\sqrt{x})a} \le \frac{|a - x|}{a\sqrt{\frac{a}{2}}} < \frac{\sqrt{2}\delta}{a^{\frac{3}{2}}} \le \frac{\sqrt{2}}{a^{\frac{3}{2}}} \cdot \frac{a^{\frac{3}{2}}\varepsilon}{\sqrt{2}} = \varepsilon. \end{aligned}$$

2. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is differentiable at $a \in I$. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $a \in \mathbf{R}$. Using just your definition of derivative and neither differentiation theorems nor chain rule, prove that $g(x) = f(x)^3$ is differentiable at a, and find $g'(a)$.*

   DEFINITION: f is differentiable at a if the real limit, call it $f'(a)$, exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

   Suppose that f is differentiable at a. We show that the difference quotient for g has a finite limit and determine its value. For $x \in I$ such that $x \neq a$,

$$\frac{g(x) - g(a)}{x - a} = \frac{f(x)^3 - f(a)^3}{x - a} = \frac{f(x) - f(a)}{x - a} \cdot \left[f(x)^2 + f(x)f(a) + f(a)^2 \right]$$

   Use the assumption that f is differentiable at $x = a$ implies that f is continuous at a so that $f(x) \to f(a)$ as $x \to a$. Now by the main theorem for limits,

$$\begin{aligned} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} \left[f(x)^2 + f(x)f(a) + f(a)^2 \right] \\ &= f'(a) \left[f(a)^2 + f(a)^2 + f(a)^2 \right] = 3f(a)^2 f'(a). \end{aligned}$$

   Thus a real limit exists and equals what we expected $g'(a) = 3f(a)^2 f'(a)$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) STATEMENT. *Let $f : [0,1] \to \mathbf{R}$ be continuous. Suppose $\{x_n\}$ is a sequence in $[0,1]$ such that $f(x_n) \to \sup_{x\in[0,1]} f(x)$ as $n \to \infty$. Then $\{x_n\}$ converges.*

FALSE. Let $f(x) = (2x-1)^2$. Then $1 = \sup_{x\in[0,1]} f(x) = f(0) = f(1)$ so f has two maximizing points. Now choose x_n to alternate between these points $x_n = \frac{1}{2}\left[(-1)^n + 1\right]$. Then $f(x_n) = 1$ converges to the supremum of f but $\{x_n\}$ does not converge.

Here $\{x_n\}$ is a maximizing sequence. The fact that a continuous function on a closed bounded interval is bounded and and takes its maximum is proved by choosing a convergent subsequence of a maximizing sequence. But if there are two maximum points, the maximizing sequence itself may not converge.

(b) STATEMENT. *If $f, g : [0,1] \to \mathbf{R}$ are differentiable, $f(0) = g(0)$ and $f'(x) > g'(x)$ for all x, then $f(x) > g(x)$ for $0 < x \le 1$.*

TRUE. Let $h(x) = f(x) - g(x)$ so $h(0) = 0$. Then for any $a \in (0,1]$ h is continuous on $[0,a]$ and differentiable on $(0,a)$. Because $h'(x) = f'(x) - g'(x) > 0$ for all $x \in (0,a)$, $h(x)$ is strictly increasing so $h(a) > 0$. It follows that $f(a) > g(a)$ as claimed. To see this, apply the Mean Value Theorem. There is $c \in (0,a)$ so that

$$h(a) = h(0) + h'(c)(a-0) = 0 + h'(c)a > 0.$$

(c) STATEMENT. *Suppose $f : (0,1) \to \mathbf{R}$ is uniformly continuous. Then f is bounded.*

TRUE. Since f is uniformly continuous on a bounded interval, it admits a continuous extension $\bar{f}$ to the closed interval $[0,1]$. The continuous function $\bar{f}$ on a closed, bounded interval is bounded, hence f is bounded.

Several people argued by contradiction and tried to shaw that if f is unbounded then f could not have been uniformly continuous, which was our homework problem 73[7]. Here is how that argument goes. The negation of uniform continuity is

DEFINITION. The function : $(0,1) \to \mathbf{R}$ is *not uniformly continuous* if there is an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are $x, y \in (0,1)$ such that $|x-y| < \delta$ but $|f(x) - f(y)| \ge \varepsilon_0$.

Assume f is unbounded. Let $x_1 \in (0,1)$ be any number. Then choose a sequence as follows. Assume $x_1, \ldots, n_n$ have been chosen. Then from unboundedness there is an $x_{n+1} \in (0,1)$ so that $|f(x_{n+1})| > 1 + |f(x_n)|$. It follows for $j > i$ from the reverse triangle inequality, using $|f(x_k)| - |f(x_{k-1})| > 1$ for all k that

$$\begin{aligned}
&|f(x_j) - f(x_i)| \ge \big||f(x_j)| - |f(x_i)|\big| \\
&= \big|(|f(x_j)| - |f(x_{j-1})|) + (|f(x_{j-1})| - |f(x_{j-2})|) + \cdots + (|f(x_{i+1})| - |f(x_i)|)\big| \\
&= (|f(x_j)| - |f(x_{j-1})|) + (|f(x_{j-1})| - |f(x_{j-2})|) + \cdots + (|f(x_{i+1})| - |f(x_i)|) \\
&> 1 + 1 + \cdots + 1 = j - i \ge 1.
\end{aligned} \tag{2}$$

Now $\{x_n\} \subset (0,1)$ so it is a bounded sequence. By the Bolzano Weierstrass Theorem, it has a convergent subsequence $\{x_{n_k}\}$ which is therefore a Cauchy Subsequence. Let $\epsilon_0 = 1$. For any $\delta > 0$ there is a $K \in \mathbf{R}$ so that

$$|x_{n_j} - x_{n_i}| < \delta \qquad \text{whenever } j, i > K.$$

Take any two natural numbers $j > i > K$. For these we have $|x_{n_j} - x_{n_i}| < \delta$ and by (2),

$$|f(x_{n_j}) - f(x_{n_i})| \ge 1 \ge \varepsilon_0.$$

Thus f is not uniformy continuous.

Here is a third (direct) argument. By uniform continuity, for $\varepsilon = 1$ there is a $\delta > 0$ so that

$$|f(x) - f(y)| < 1 \qquad \text{whenever } x, y \in (0,1) \text{ and } |x - y| < \delta. \tag{3}$$

By the Archimedean Property there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Then there is the bound

$$|f(x)| \le n + \left| f\left(\frac{1}{2}\right) \right| \qquad \text{for all } x \in (0,1). \tag{4}$$

To see this, choose $a \in (0,1)$. consider the points

$$y_k = \frac{k}{2n} + \frac{(n-k)a}{n}, \qquad \text{for } k = 1, \ldots, n.$$

These points are equally spaced between a and $\frac{1}{2}$ whose distance apart is $|y_{k-1} - y_k| < \frac{1}{n} < \delta$. Thus, using (3),

$$\begin{aligned} \left| f(a) - f\left(\frac{1}{2}\right) \right| &= |(f(y_0) - f(y_1)) + (f(y_1) - f(y_2)) + \cdots + (f(y_{n-1}) - f(y_n))| \\ &\le |f(y_0) - f(y_1)| + |f(y_1) - f(y_2)| + \cdots + |f(y_{n-1}) - f(y_n)| \\ &< 1 + 1 + \cdots + 1 = n. \end{aligned}$$

Hence (4) follows

$$|f(a)| = \left| \left\{ f(a) - f\left(\frac{1}{2}\right) \right\} + f\left(\frac{1}{2}\right) \right| \le \left| f(a) - f\left(\frac{1}{2}\right) \right| + \left| f\left(\frac{1}{2}\right) \right| \le n + \left| f\left(\frac{1}{2}\right) \right|.$$

4. *Let $\mathcal{D} \subset \mathbf{R}$ and $f, f_n : \mathcal{D} \to \mathbf{R}$ be functions. State the definitions:*

   (a) $\{f_n(x)\}$ converges pointwise *to a function f on $\mathcal{D}$ as $n \to \infty$.*

   (b) $\{f_n(x)\}$ converges uniformly *to a function f on $\mathcal{D}$ as $n \to \infty$.*

   Determine whether the functions $f_n(x) = \dfrac{1}{1 + x^n}$ converge pointwise, converge uniformly, or do not converge to a function $f(x)$ on $(0, \infty)$ and prove your result.

   DEFINITIONS.

   (a) $\{f_n(x)\}$ *converges pointwise* to a function f on $\mathcal{D}$ as $n \to \infty$ if for every $x \in \mathcal{D}$ and every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that

   $$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n > N.$$

   (b) $\{f_n(x)\}$ *converges uniformly* to a function f on $\mathcal{D}$ as $n \to \infty$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that

   $$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n > N \text{ and } x \in \mathcal{D}.$$

   $f_n(x) = \dfrac{1}{1 + x^n}$ converge pointwise to $f(x)$ on $(0, \infty)$ where

   $$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1; \\ \frac{1}{2}, & \text{if } x = 1; \\ 0, & \text{if } 1 < x. \end{cases} \qquad \text{since} \qquad \lim_{n \to \infty} \frac{1}{1 + x^n} = \begin{cases} \frac{1}{1+0}, & \text{if } 0 < x < 1; \\ \frac{1}{1+1}, & \text{if } x = 1; \\ \frac{1}{1+\infty}, & \text{if } 1 < x. \end{cases}$$

   The convergence is not uniform. The functions $f_n(x)$ are continuous on $(0, \infty)$. If the convergence were uniform, then the uniform limit of continuous functions would have to be continuous, however, the limiting function here, which would have to be the same as the pointwise limiting function $f(x)$, is not continuous at $x = 1$.

Another argument is to consider the sequence $\{x_n\} \subset (0, \infty)$ given by

$$x_n = 2^{\frac{1}{n}}.$$

For this sequence

$$f_n(x_n) - f(x_n) = \frac{1}{3}$$

for all n, which is not tending to zero as $n \to \infty$ which would be the case were the convergence uniform.

5. *Let $f : \mathbf{R} \to \mathbf{R}$ is continuous and $f(r) = 0$ at each rational number $r \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbf{R}$.*

We prove that $f(a) = 0$ for any real number $a \in \mathbf{R}$. The easiest way is to use the sequential characterization of continuity. Since f is continuous at a, for every real sequence $\{x_i\} \subset \mathbf{R}$ sucn that $x_n \to a$ as $n \to \infty$ we have

$$f(a) = \lim_{n\to\infty} f(x_n).$$

Now, the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, thus we can choose a sequence of rational numbers $r_n \in \mathbb{Q}$ such that $r_n \to a$ as $n \to \infty$ To see this, for every $n \in \mathbb{N}$, by the density of rationals, there is a rational number in every open interval, so we choose a rational $r_n \in (a, a + \frac{1}{n})$. Since $|a - r_n| < \frac{1}{n}$ we have $r_n \to a$ as $n \to \infty$. Applying the sequential characterization to this sequence,

$$f(a) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0$$

because, at rational numbers $f(r_n) = 0$.

An alternative argument uses only the definition of continuity. Choose $a \in \mathbf{R}$. Since f is continuous at a, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(a) - f(x)| < \varepsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } |x - a| < \delta.$$

By the density of rationals, there is a rational number r such that $|r - a| < \delta$. Hence for this r

$$|f(a) - f(r)| < \varepsilon.$$

However, since f vanishes at rational numbers, $f(r) = 0$. Thus

$$|f(a)| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we conclude that $f(a) = 0$.