Math 3210 § 1. **Second Midterm Exam** Name:______________
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1. *Let* $a_n = \sqrt{\dfrac{n-1}{n+1}}$ *. Define:* $L = \lim_{n\to\infty} a_n$*. Find* $L$ *using limit laws.*
*Prove using just your definition that* $L = \lim_{n\to\infty} a_n$*.*

A real number $L$ is the limit $L = \lim_{n\to\infty} a_n$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N.$$

Using the root law, quotient law and sum law,

$$L = \lim_{n\to\infty} \sqrt{\frac{n-1}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{n-1}{n+1}}$$
$$= \sqrt{\lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}}} = \sqrt{\frac{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1-0}{1+0}} = 1.$$

To prove that $\lim_{n\to\infty} a_n = 1$, choose $\epsilon > 0$. Let $N = \dfrac{2}{\varepsilon} - 1$. For any $n \in \mathbb{N}$ such that $n > N$ we have

$$|a_n - 1| = \left|\sqrt{\frac{n-1}{n+1}} - 1\right| = \left|\left(\sqrt{\frac{n-1}{n+1}} - 1\right)\frac{\sqrt{\frac{n-1}{n+1}}+1}{\sqrt{\frac{n-1}{n+1}}+1}\right| = \left|\frac{\frac{n-1}{n+1} - 1}{\sqrt{\frac{n-1}{n+1}}+1}\right|$$
$$\le \left|\frac{n-1}{n+1} - 1\right| = \left|\frac{n-1}{n+1} - \frac{n+1}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{N+1} = \varepsilon.$$

2. *Define the real sequence* $\{a_n\}$ *recursively by* $a_1 = 1$ *and by* $a_{n+1} = 6 + \sqrt{a_n}$ *for* $n \ge 1$*. Show that* $\{a_n\}$ *is convergent.*

We show that $\{a_n\}$ is increasing and bounded above. Thus, by the Monotone Convergence Theorem, $a_n \to a$ as $n \to \infty$ for some $a \in \mathbf{R}$.

To show that $\{a_n\}$ is increasing, note that $f(x) = 6 + \sqrt{x}$ is increasing on $[0, \infty)$. Argue by induction to show that $1 \le a_n < a_{n+1}$.

For the base case, we see that

$$a_2 = f(a_1) = 6 + \sqrt{a_1} = 6 + \sqrt{1} = 7 > 1 = a_1.$$

For the induction case, assume for some $n$ that $1 \le a_n < a_{n+1}$. $1 \le a_{n+1}$ is immediate. Applying $f$ we see that

$$a_{n+2} = f(a_{n+1}) > f(a_n) = a_{n+1}$$

since $f$ is increasing and both $a_n$ and $a_{n+1}$ are in the domain of $f$. Thus it follows by induction that $1 \le a_n < a_{n+1}$ for all $n$.

To show that $\{a_n\}$ is bounded above, we shall show thqt $a_n \le 9$. In fact any larger number will work also. Arguing by induction, the base case follows since we are given $a_1 = 1 < 9$.

For the induction case, assume that $a_n \le 9$ for some $n$. Then from before $a_n \ge 1$ so $a_n$ is in the domain of $f$ and

$$a_{n+1} = 6 + \sqrt{a_n} \le 6 + \sqrt{9} = 9.$$

Thus it follows by induction that $a_n \le 9$ for all $n$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT. *Let $I_n$ be a sequence of bounded intervals such that $I_1 \supset I_2 \supset \cdots$. Then* $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

   FALSE. If the intervals were closed then the answer would have been "true" by the Nested Intervals Theorem. But being closed was not specified, so if we take $I_n = \left(0, \frac{1}{n}\right)$ then $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

   (b) STATEMENT. *No real sequence $\{a_n\}$ satisfies* $\limsup_{n\to\infty} a_n = -\infty$.

   FALSE. Take the sequence $a_n = -n$ which converges to $-\infty$. For the lim sup, we note that

$$s_n = \sup\{a_k : k \geq n\} = -n$$

   so that

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n = -\infty.$$

   (c) STATEMENT. *Suppose the real sequence $\{a_n\}$ is not bounded above. Then there is a subsequence $a_{n_k} \to \infty$ as $k \to \infty$.*

   TRUE. Since the sequence is not bounded above, for every $M \in \mathbf{R}$ there is an $n$ such that $a_n > M$. We select a subsequence of larger and larger terms. The only technicality is to arrange that the terms occur in increasing order in the sequence. Start by choosing $n_1 \in \mathbb{N}$ so that $a_{n_1} > 1$. Then choose $n_2$ so that

$$a_{n_2} > \max\{a_1, \ldots, a_{n_1}, 2\}.$$

   Since $a_{n_2}$ is larger than all $a_1, \ldots, a_{n_1}$ the $n_2$ cannot be any of $1, 2, \ldots, n_1$ thus $n_2 > n_1$. Also $a_{n_2} > 2$. Continue in this fashion. Suppose that $n_1 < \ldots < n_j$ have been chosen such that $a_{n_j} > j$. Then choose $n_{j+1} \in \mathbb{N}$ so that

$$a_{n_{j+1}} > \max\{a_1, \ldots, a_{n_j}, j+1\}$$

   Since $a_{n_{j+1}}$ is larger than all $a_1, \ldots, a_{n_j}$ the $n_{j+1}$ cannot be any of $1, 2, \ldots, n_j$, thus it must satisfy $n_{j+1} > n_j$. Also $a_{n_{j+1}} > j+1$.

   Thus we have constructed a subsequence $a_{n_j} > j$ which tends to $\infty$ as $j \to \infty$.

4. *Let $\{a_b\}$ and $\{b_n\}$ be a real sequences which converge to real numbers $a_n \to a$ and $b_n \to b$ as $n \to \infty$ and that for some $N \in \mathbf{R}$,*

$$a_n \ \le\ b_n \qquad \textit{whenever } n > N.$$

*Using just the definition of convergence, prove that $a \le b$.*

We show that for every $\varepsilon > 0$ we have $b - a > -\varepsilon$ which implies $b - a \ge 0$. Choose $\varepsilon > 0$. Since $a_n \to a$ and $b_n \to b$ as $n \to \infty$, there are $N_1$ and $N_2$ in $\mathbf{R}$ so that

$$|a_n - a| < \frac{\varepsilon}{2}, \qquad \text{whenever } n > N_1;$$
$$|b_n - b| < \frac{\varepsilon}{2}, \qquad \text{whenever } n > N_2.$$

By the Archimedean Property, there is $n \in \mathbb{N}$ such that $n > \max\{N_1, N_2, N\}$. For this $n$

$$b - a = b_n + (b - b_n) - a_n - (a - a_n) \ge b_n - a_n - |b - b_n| - |a - a_n| > 0 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon.$$

We have shown that $b - a > -\varepsilon$ for every $\varepsilon > 0$ which implies $b - a \ge 0$.

5. *Define: $\{x_n\}$ is a* Cauchy Sequence. *Let* $x_n = \displaystyle\sum_{k=1}^{n} \frac{1 - 2\cos(k)}{k!}$.

*Prove that $\{x_n\}$ is convergent.*

A real sequence $\{x_n\}$ is a Cauchy Sequence if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|x_m - x_\ell| < \varepsilon, \qquad \text{whenever } m > N \text{ and } \ell > N.$$

Observe that

$$|1 - 2\cos(k)| \le |1| + |2\cos(k)| \le 1 + 2 = 3. \tag{1}$$

Also, the factorial satisfies

$$k! \ge 2^{k-1} \tag{2}$$

for all $k \in \mathbb{N}$. We can see this by induction. For the base case $1! = 1 = 2^{1-1}$. For the induction case, assume that $k! \ge 2^{k-1}$ for some $k \in \mathbb{N}$. Then since $k \ge 1$,

$$(k+1)! = (k+1) \cdot k! \ge 2 \cdot 2^{k-1} = 2^{(k+1)-1}.$$

Hence by induction, $k! \ge 2^{k-1}$ for all $k \in \mathbb{N}$.

To prove that $\{x_n\}$ converges we show that it is a Cauchy Sequence, hence convergent. Choose $\varepsilon > 0$. Let $N \in \mathbf{R}$ be such that $\dfrac{3}{2^{N-1}} = \varepsilon$. Suppose that $m, \ell \in \mathbb{N}$ such that $m > N$ and $\ell > N$. Then if $m = \ell$ we have $|x_m - x_\ell| = 0 < \varepsilon$. If $m \ne \ell$, without loss of generality we may assume $m > \ell$. Otherwise, we may swap the roles of $m$ and $\ell$. We have by the triangle inequality, (1), (2) and replacing the dummy index by $k = \ell + 1 + j$,

$$|x_m - x_\ell| = \left| \sum_{k=1}^{m} \frac{1 - 2\cos(k)}{k!} - \sum_{k=1}^{\ell} \frac{1 - 2\cos(k)}{k!} \right| = \left| \sum_{k=\ell+1}^{m} \frac{1 - 2\cos(k)}{k!} \right| \le \sum_{k=\ell+1}^{m} \frac{|1 - 2\cos(k)|}{k!}$$

$$\le \sum_{k=\ell+1}^{m} \frac{3}{k!} \le \sum_{k=\ell+1}^{m} \frac{3}{2^{k-1}} = \frac{3}{2^\ell} \sum_{j=0}^{m-\ell-1} \frac{1}{2^j} = \frac{3}{2^\ell} \cdot \frac{1 - \frac{1}{2^{m-\ell}}}{1 - \frac{1}{2}} < \frac{3}{2^{\ell-1}} < \frac{3}{2^{N-1}} = \varepsilon.$$

We have used the formula for the sum of a geometric series

$$\sum_{j=0}^{p} r^j = \frac{1 - r^{p+1}}{1 - r}.$$