Math 3210 § 2. **Third Midterm Exam** Name: Solutions
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1. *Let $f_n, f : [0,1] \to \mathbf{R}$. Define: $f_n \to f$ uniformly on $[0,1]$. Let $f_n(x) = \dfrac{nx}{nx^2+n+1}$. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$. Determine whether this limit $f_n \to f$ on $[0,1]$ is uniform and prove your assertion.*

   $f_n \to F$ uniformly on $\mathbf{R}$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that
$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } n > N.$$

   The pointwise limit at $x \in \mathbf{R}$ exists and equals
$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{nx^2+n+1} = \lim_{n\to\infty} \frac{x}{x^2+1+\frac{1}{n}} = \frac{x}{x^2+1}.$$

   The convergence is unform on $\mathbf{R}$. To see this, choose $\varepsilon > 0$ and let $N = \dfrac{1}{2\varepsilon}$. For any $x \in \mathbf{R}$ and any $n > N$ we have
$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{nx}{nx^2+n+1} - \frac{x}{x^2+1}\right| = \left|\frac{nx(x^2+1) - x(nx^2+n+1)}{(nx^2+n+1)(x^2+1)}\right| \\ &= \frac{|-x|}{(nx^2+n+1)(x^2+1)} \le \frac{|x|}{n(x^2+1)^2} \le \frac{1}{2n(x^2+1)} \le \frac{1}{2n} < \frac{1}{2N} = \varepsilon,\end{aligned}$$
   where we have used that $0 \le (|x|-1)^2 = (x^2+1) - 2|x|$.

2. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$. State the definition: $f(x)$ is differentiable at $a \in I$. Suppose $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ satisfies $|f(x)| \le |x-a|^2$ for all x. Using just your definition of derivative and properties of limits of functions, prove carefully that f is differentiable at a, and find $f'(a)$.*

   $f(x)$ is differentiable at $a \in I$ means that the following limit exists as a real number
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}.$$

   We note that the condition on f implies that f lies between the two parabolas: for all x,
$$-|x-a|^2 \le f(x) \le |x-a|^2$$
   so $|f(a)| \le |a-a|^2 = 0$ or $f(a) = 0$. Thus the difference quotient satisfies
$$\begin{aligned}-|x-a| = -\frac{|x-a|^2}{|x-a|} &\le -\frac{|f(x)|}{|x-a|} = -\left|\frac{f(x)-f(a)}{x-a}\right| \le \frac{f(x)-f(a)}{x-a} \\ &\le \left|\frac{f(x)-f(a)}{x-a}\right| = \frac{|f(x)|}{|x-a|} \le \frac{|x-a|^2}{|x-a|} = |x-a|.\end{aligned}$$

   Thus by letting $x \to a$ we see that both ends tend to zero. Thus by the squeeze theorem, the middle limit exists and equals zero, namely the function is differentiable at a and
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0.$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Suppose $f : [0,1] \to \mathbf{R}$ is an integrable function. Then* $F(x) = \int_0^x f(t)\,dt$ *is differentiable at* $a = \frac{1}{2}$.

   FALSE. Let

$$f(x) = \begin{cases} -1, & \text{if } x \le \frac{1}{2}; \\ 1, & \text{if } x > \frac{1}{2}. \end{cases}$$

   Then $F(x) = |x - \frac{1}{2}| - \frac{1}{2}$ which has a kink at $x = \frac{1}{2}$ and thus is not differentiable there.

   (b) STATEMENT: *Let $y = f(x)$ be a differentiable and strictly increasing function defined on the real numbers such that $f(a) = b$ for some $a, b \in \mathbf{R}$. Then the inverse function $x = g(y)$ is differentiable at b.*

   FALSE. The function $f(x) = x^3$ is differentiable and strictly increasing on $\mathbf{R}$. The inverse function is $g(y) = y^{1/3}$. At corresponding points $a = b = 0$, $f(a) = b$ but g is not differentiable at b. The theorem about the existence of the derivative of an inverse function cannot be applied at $a = 0$ because $f'(a) = 0$.

   (c) STATEMENT: *Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function. If $f'(x) > 0$ at all x then f is strictly increasing.*

   TRUE. Let $x, y \in \mathbf{R}$ such that $x < y$. Then f is continuous on $[x, y]$ because it is differentiable on all of $\mathbf{R}$, and so differentiable on (x, y). Thus the Mean Value Theorem applies: there is $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x) > 0$$

   because $f'(c) > 0$ by assumption and $y - x > 0$ by choice of x and y. This holds for all $x < y$ so f is increasing.

4. *Let f be a bounded function on the closed bounded interval $[a, b]$. Define what it means for f to be integrable on $[a, b]$ and what the Riemann integral of f on $[a, b]$ is. Find the upper integral* $\overline{\int_0^2} f(x)\,dx$ *and the lower integral* $\underline{\int_0^2} f(x)\,dx$ *where*

$$f(x) = \begin{cases} \pi, & \text{if } x \text{ is rational and } x > 1; \\ \\ 0, & \text{otherwise.} \end{cases}$$

   What does this say about f?

   The lower and upper integrals of f on $[a, b]$ are defined by

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(f, p), \qquad \overline{\int_a^b} f(x)\,dx = \inf_P U(f, p),$$

   where $P = \{a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b\}$ are partitions of $[a, b]$ and the lower and upper sums are defined by

$$L(f, P) = \sum_{i=1}^n m_i (x_i - x_{i-1}), \qquad U(f, P) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$

   where for each subinterval $I_i = [x_{i-1}, x_i]$ the infimum and supermum are defined by

$$m_i = \inf_{I_i} f, \qquad M_i = \sup_{I_i} f.$$

f is said to be Rienann Integrable on $[a, b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx,$$

and their common value is the Riemann Integral $\displaystyle\int_a^b f(x)\,dx$.

In case of this function

$$f(x) = \begin{cases} \pi, & \text{if } x \text{ is rational and } x > 1; \\ \\ 0, & \text{otherwise.} \end{cases}$$

we see that $f(z) = 0$ for irrational z. The irrationals are dense in $[0, 2]$ so the infimum of f is zero on all nonzero length intervals so $m_j = 0$ if $x_{j-1} < x_j$. Similarly, we see that $M_i = \pi$ on all nonzero length intervals that intersect $[1, 2]$ since the rationals are dense in $[1, 2]$. For subintervals of $I_k \subset [0, 1)$ the function is zero so $M_k = 0$. Thus if P is a partition of $[0, 2]$, let q be the index that satisfies $x_{q-1} < 1 \le x_q$. Then the upper and lower sums are

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \widetilde{\sum_{j=1}^{n}} m_j(x_j - x_{j-1}) = 0; \\ U(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{k=1}^{q-1} M_k(x_k - x_{k-1}) + \sum_{i=q}^{n} M_i(x_i - x_{i-1}) \\ &= 0 + \widetilde{\sum_{j=q}^{n}} M_j(x_j - x_{j-1}) = \widetilde{\sum_{j=q}^{n}} \pi(x_j - x_{j-1}) = \pi(2 - x_{q-1}) \end{aligned}$$

where $\widetilde{\sum}$ denotes the sum over nonzero width intervals only, the zero width ones are excluded from this sum since they contribute zero to the sum. We conclude that

$$\underline{\int_0^2} f(x)\,dx = \sup_P L(f, P) = 0, \qquad \overline{\int_0^2} f(x)\,dx = \inf_P U(f, P) = \pi,$$

since $1 - x_{q-1} > 0$ may be arbitrarily small in the choice of P. Since lower and upper integrals differ, the function is not Riemann integrable on $[0, 2]$.

5. *Let f be a bounded function on the closed bounded interval $[a, b]$. Complete the statement of the theorem. [Of several possible answers, select the one you prefer to answer the last question.]*

   Theorem. *The bounded function f is integrable on $[a, b]$ if and only if*

   there is a sequence of partitions P_n of $[a, b]$ such that

   $$U(f, P_n) - L(f, P_n) \to 0 \qquad \text{as } n \to \infty.$$

   Using only your theorem, show that if $g(x)$ is integrable on $[0, 1]$, where

   $$g(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ -2, & \text{if } x = 1. \end{cases}$$

Let P_n be the partition with n equally spaced intervals, *i.e.*, $x_i = \frac{i}{n}$. Let $I_i = [x_{i-1}, x_i]$ and so $x_i - x_{i-1} = \frac{1}{n}$. Since f is increasing in $[0,1)$, for all $i = 0, \ldots, n-1$ we have

$$m_i = \inf_{I_i} f = x_{i-1}, \qquad M_i = \sup_{I_i} f = x_i.$$

In the last interval $f(1) = -2$ so the infimum and supremum are

$$m_n = \inf_{I_n} f = -2, \qquad M_n = \sup_{I_n} f = 1.$$

Now we can compute the difference of upper and lower sums

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = \left(\sum_{i=1}^{n-1} (x_i - x_{i-1}) \frac{1}{n} \right) + (1 - (-2)) \frac{1}{n} \\ &= \frac{x_{n-1} - x_0}{n} + \frac{3}{n} = \frac{n-1}{n^2} + \frac{3}{n} \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Thus by the theorem, f is integrable on $[0,1]$.