Math 3210 § 2. **First Midterm Exam** Name: Solutions

Treibergs January 28, 2026.

1. *Prove that for all $n \in \mathbb{N}$,*

$$P(n) \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Argue by induction. For the base case $n = 1$ we have

$$\text{Lhs.} = \sum_{k=1}^{1} k^3 = i^3 = 1, \qquad \text{Rhs.} = \frac{1^2(1+1)^2}{4} = 1,$$

which are equal, so $P(1)$ holds.

For the induction case, we assume that $P(n)$ holds for some $n \in \mathbb{N}$ (the induction hypothesis). Consider the left side for $n+1$. Using the induction hypothesis

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \left(\sum_{k=1}^{n} k^3\right) + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2\left(\frac{n^2}{4} + n + 1\right) \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4},\end{aligned}$$

which show that then $P(n+1)$ holds. Since both the base and induction cases hold, we conclude by induction that $P(n)$ holds for all $n \in \mathbb{N}$.

2. *Recall the axioms of a field $(\mathcal{F}, +, \times)$. For any $x, y, z \in \mathcal{F}$,*

[A1.]	(Commutativity of Addition)	$x + y = y + x$.
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z$.
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\,(\forall\, t \in \mathcal{F})\ 0 + t = t$.
[A4.]	(Additive Inverse)	$(\exists\, -x \in \mathcal{F})\ x + (-x) = 0$.
[M1.]	(Commutativity of Multiplication)	$xy = yx$.
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z$.
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})\ 1 \neq 0$ and $(\forall\, t \in \mathcal{F})\ 1t = t$.
[M4.]	(Multiplicative Inverse)	If $x \neq 0$ then $(\exists\, x^{-1} \in \mathcal{F})\ (x^{-1})x = 1$.
[D.]	(Distributivity)	$x(y + z) = xy + xz$.

Using only the field axioms, show that if $a, b \in F$ then $(a + b)^2 = a^2 + 2ab + b^2$. Justify every step of your argument using just the axioms listed here. Do not quote any formulas from the text.

$$
\begin{aligned}
(a+b)^2 &= (a+b)(a+b) && \text{Start from left side. Meaning of square.}\\
&= (a+b)a + (a+b)b && \text{By distributive (D).}\\
&= a(a+b) + b(a+b) && \text{By commutativity of multiplication (M1).}\\
&= (a^2+ab) + (ba+b^2) && \text{By distrubutive (D).}\\
&= [(a^2+ab)+ba] + b^2 && \text{By associativity of addition (A2).}\\
&= [a^2+(ab+ba)] + b^2 && \text{By associativity of addition (A2).}\\
&= [a^2+(ab+ab)] + b^2 && \text{By commutativity of multiplication (M1).}\\
&= [a^2+(1\{ab\}+1\{ab\})] + b^2 && \text{Multiplicative identity.}\\
&= [a^2+(\{ab\}1+\{ab\}1)] + b^2 && \text{By commutativity of multiplication (M2).}\\
&= [a^2+(\{ab\}[1+1])] + b^2 && \text{By distributive (D).}\\
&= [a^2+\{ab\}2] + b^2 && 1+1=2.\\
&= [a^2+2\{ab\}] + b^2 && \text{By commutativity of multiplication (M2).}\\
&= a^2+[2\{ab\} + b^2] && \text{By associativity of addition (A2).}
\end{aligned}
$$

Because the last two expressions are equal, there is no ambiguity in concluding $(a+b)^2 = a^2 + 2\{ab\} + b^2$ because both ways of grouping the addition in the right side sum yield the same result. By the same token by associativity of multiplication $2\{ab\} = \{2a\}b$, there is no ambiguity in concluding $2\{ab\} = 2ab$ because both ways of grouping the product yield the same result. So the right side equals $a^2 + 2ab + b^2$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let $f : \mathbf{R} \to \mathbf{R}$ and $E, G \subset \mathbf{R}$. Then $f(E \backslash G) = f(E) \backslash f(G)$.*
   FALSE. Consider $f(x) = x^2$, $E = [-2, 3]$, $G = [1, 3]$ then $E \backslash G = [-2, 1)$ so $f(E \backslash G) = [0, 4]$ but $f(E) = [0, 9]$ and $f(G) = [1, 9]$ so $f(E) \backslash f(G) = [0, 1)$ which is not the same as $f(E \backslash G)$.

   (b) STATEMENT: *If $f : \mathbf{R} \to \mathbf{R}$ is onto and $V, W \subset \mathbf{R}$ such that $f^{-1}(V) \cap f^{-1}(W) = \emptyset$ then $V \cap W = \emptyset$.*
   TRUE. Argue by contrapositive. Assume the conclusion is false $V \cap W \neq \emptyset$. Hence there is a $y \in V \cap W$. But f is onto so y is in the range of f so that $f^{-1}(V \cap W) \neq \emptyset$. Using the identity for pullpack, $f^{-1}(V) \cap f^{-1}(W) = f^{-1}(V \cap W) \neq \emptyset$. Thus the hypothesis is false, proving the assertion.

   (c) STATEMENT: $S = \Big\{x \in \mathbf{R} : (\forall \varepsilon > 0)(\exists t > 1) x + t < \varepsilon\Big\} = (-\infty, -1]$.
   Arguing using intersections and intersections,
   $$S = \bigcap_{\varepsilon>0} \bigcup_{t>1} (-\infty, \varepsilon - t) = \bigcap_{\varepsilon>0} (-\infty, \varepsilon - 1) = (-\infty, -1].$$
   Arguing by set inclusion, to show $(-\infty, -1] \subset S$ choose $x \in (-\infty, -1]$. Then $x \leq -1$. No matter which $\varepsilon > 0$ is chosen, by taking $t = 1 + \varepsilon/2 > 1$ we have $x + t = x + 1 + \varepsilon/2 \leq -1 + 1 + \varepsilon/2 < \varepsilon$. Thus $x \in S$. To show $S \subset (-\infty, -1]$ assume that $y \notin (-\infty, -1]$. Then $y > -1$. Let $\varepsilon = 1 + y > 0$ Then for any $t > 1$ we have $y + t > y + 1 \geq \varepsilon$ so that $y \notin S$.

4. *Recall that the rational numbers are defined to be the set of equivalence classes $\mathbb{Q} = S/\sim$ where $S = \left\{\dfrac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\right\}$ is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction: $\dfrac{a}{b} \sim \dfrac{c}{d}$ iff $ad = bc$. We denote the*

equivalence class, the "fraction" by $\left[\frac{a}{b}\right]$ *to distinguish it from a symbol from* S*. Addition and multiplication of rationals, for example, is defined on equivalence classes by*

$$\left[\frac{m}{n}\right]+\left[\frac{r}{t}\right]=\left[\frac{mt+nr}{nt}\right], \qquad \left[\frac{m}{n}\right]\cdot\left[\frac{r}{t}\right]=\left[\frac{mr}{nt}\right].$$

How is $x \le y$ *defined in the rational numbers? Show that the function* $f : \mathbb{Q} \to \mathbb{Q}$ *is well defined, where*

$$f\left(\left[\frac{a}{b}\right]\right)=\left[\frac{a^2-b^2}{a^2+b^2}\right]$$

To define $x \ge 0$ for a rational

$$x=\left[\frac{a}{b}\right]$$

the condition is equivalent to $ab \ge 0$ in the integers. Also, you can always pick a representative in which $b > 0$ and then the condition for $x \ge 0$ reduces to $a \ge 0$. The definition of $x \le y$ is that $y - x \ge 0$. In terms of fractions,

$$x=\left[\frac{a}{b}\right], \quad y=\left[\frac{c}{d}\right], \quad y-x=\left[\frac{cb-da}{db}\right]\ge 0 \text{ means } (cb-da)bd \ge 0.$$

To show $f(x)$ is well defined, we need to show that if x is represented by an equivalent rational, then the value of f works out to be equivalent when computed in the new representative. Let us suppose

$$\frac{a}{b}\sim\frac{c}{d} \text{ or } ad = bc.$$

Then compute f using two reepresentatives. Is

$$f\left(\left[\frac{a}{b}\right]\right)=\left[\frac{a^2-b^2}{a^2+b^2}\right]\overset{?}{=}\left[\frac{c^2-d^2}{c^2+d^2}\right]=f\left(\left[\frac{c}{d}\right]\right),$$

which is equivalent to is

$$\frac{a^2-b^2}{a^2+b^2}\sim\frac{c^2-d^2}{c^2+d^2} \quad \text{or} \quad (a^2-b^2)(c^2+d^2)=(a^2+b^2)(c^2-d^2)? \tag{1}$$

Using $ad = bc$ we see that

$$\begin{aligned}(a^2-b^2)(c^2+d^2) &= a^2c^2+a^2d^2-b^2c^2-b^2d^2\\ &= a^2c^2+b^2c^2-a^2d^2-b^2d^2\\ &= (c^2-d^2)(a^2+b^2)\end{aligned}$$

which shows that (1) holds so f is well defined.

5. *Let* $E \subset \mathbf{R}$ *be a nonempty subset which is bounded above. State the definition:* $\ell = \operatorname{lub} E$*. Find* $\ell = \operatorname{lub} E$ *and prove your answer, where* $E = \left\{\frac{n^2-1}{3n^2+1} : n \in \mathbb{N}\right\}$.

The number $\ell \in \mathbf{R}$ is the least upper bound of E if it is both an upper bound and the least among upper bounds. In other words (1) $x \le \ell$ for all $x \in E$; and (2), if b is an upper bound for E then $\ell \le b$. Another way to formulate (2) is to assert no number less than ℓ is an upper bound, in other words, (2′) $(\forall \varepsilon > 0)(\exists x \in E)(\ell - \varepsilon < x)$.

First we claim that $\ell = \frac{1}{3}$ is an upper bound. Indeed, for any $n \in \mathbb{N}$ we have

$$\frac{n^2-1}{3n^2+1}=\frac{n^2-\frac{1}{3}-\frac{2}{3}}{3n^2+1}=\frac{1}{3}-\frac{2}{3(3n^2+1)}\le\frac{1}{3}.$$

Second we claim that no smaller number is an upper bound. To show (2′), choose $\varepsilon > 0$. By the Archimedean property, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{9}{2}\varepsilon.$$

Then, for this n_0 we have using $n_0^2 \geq n_0$,

$$\frac{n_0^2 - 1}{3n_0^2 + 1} = \frac{1}{3} - \frac{2}{3(3n_0^2 + 1)} > \frac{1}{3} - \frac{2}{9n_0^2} \geq \frac{1}{3} - \frac{2}{9n_0} > \frac{1}{3} - \varepsilon.$$

Thus for every $\varepsilon > 0$ there is member $x \in E$ such that $\ell - \varepsilon < x$ which proves ℓ is least among upper bounds.