Math 3210 § 2. **Sample Final Exams** Name: SomeSolutions
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More Problems.

1. *Suppose $f : [-5, \infty) \to \mathbb{R}$ is defined by $f(x) = \sqrt{5 + x}$. Using just the definition of differentiability, show that f is differentiable at $a = 5$ and find $f'(4)$.*

   We show that the limit of the difference quotient exists.

$$\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\
&= \lim_{x\to 4} \frac{\sqrt{5+x} - \sqrt{5+4}}{x-4} \\
&= \lim_{x\to 4} \frac{(\sqrt{5+x} - 3)\,(\sqrt{5+x} + 3)}{(x-4)\,(\sqrt{5+x}+3)} \\
&= \lim_{x\to 4} \frac{5 + x - 9}{(x-4)\,(\sqrt{5+x}+3)} \\
&= \lim_{x\to 4} \frac{1}{\sqrt{5+x}+3} = \frac{1}{6}.
\end{aligned}$$

2. *Suppose $f : (a, b) \to \mathbb{R}$ is uniformly continuous. Show that the limit exists: $\lim\limits_{x\to b-} f(x)$.*

   Proof. *i.e.,* a uniformly continuous function on (a, b) has a continuous extension to $(a, b]$. This was a theorem in the text, but the problem asks us to prove it. *Uniformly continuous* means for every $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in (a, b)$ satisfy $|x - y| < \delta$. Let $\{x_n\}$ be any sequence in (a, b) such that $x_n \to 0$ as $n \to \infty$. We first show that $\{f(x_n)\}$ is a Cauchy Sequence. To see it, choose $\varepsilon > 0$. By uniform continuity, there is a $\delta > 0$ so that $|f(x_m) - f(x_k)| < \varepsilon$ whenever $|x_m - x_k| < \delta$. But $\{x_n\}$ is convergent, hence Cauchy. Thus there is an $N \in \mathbb{R}$ so that $|x_m - x_k| < \delta$ whenever any $m, k \in \mathbb{N}$ satisfy $m > N$ and $k > N$. Hence $|f(x_m) - f(x_k)| < \varepsilon$ whenever any $m, k \in \mathbb{N}$ satisfy $m > N$ and $k > N$. But this says $\{f(x_n)\}$ is a Cauchy Sequence.

   Since $\{f(x_n)\}$ is Cauchy, it is convergent, so let $L \in \mathbb{R}$ be the limit: $f(x_n) \to L$ as $n \to \infty$. We have found a subsequence converging to L. The rest of the argument is to show that continuous limit $f(x) \to L$ as $x \to b-$. To this end, we show that the definition of limit is satisfied: that for all $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ for all $x \in (a, b)$ such that $b - \delta < x < b$. Choose $\varepsilon > 0$. By uniform convergence, there is a $\delta > 0$ so that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $x, y \in (a, b)$ satisfy $|x - y| < \delta$. This is the δ needed for the limit. Now choose any $x \in (a, b)$ such that $b - \delta < x < b$. Since $f(x_n) \to L$ as $n \to \infty$, there is an $N \in \mathbb{R}$ so that $|f(x_n) - L| < \frac{\varepsilon}{2}$ whenever $n > N$. Finally, since $x_k \to b$ as $k \to \infty$, there is a $k \in \mathbb{N}$ so large that $k > N$ and $b - \delta < x_k < b$. By the usual sneaky adding and subtracting trick and the triangle inequality,

$$|f(x) - L| = |f(x) - f(x_k) + f(x_k) - L| \le |f(x) - f(x_k)| + |f(x_k) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

   by uniform continuity since both $x, x_k \in (b - \delta, b)$ so $|x - x_k| < \delta$ and by the Cauchy of $f(x_n)$. Thus $f(x) \to L$ as $x \to b-$.

3. *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at $a = 0$ but not for any other $a \ne 0$. Prove that your function has this property.*

   Proof. We modify the Dirichlet function that is not continuous at any point. Let

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \text{ is rational,} \\ 0, & \text{if } \in \mathbb{R}\backslash\mathbb{Q} \text{ is irrational.} \end{cases}$$

If $a \neq 0$ then f is not continuous at a, hence not differentiable at a. Indeed by the density of rationals and irrationals, there are sequences $y_n \in \mathbb{Q}$ and $z_n \in \mathbb{R}\backslash\mathbb{Q}$ such that $y_n \to a$ and $z_n \to a$ as $n \to \infty$. Thus $f(y_n) = y_n^2 \to a^2$ as $n \to \infty$ but $f(z_n) = 0 \to 0$ as $n \to \infty$. Since two subsequences converging to a result in inconsistent limits ($a^2 \neq 0$), the function f is not continuous at a.

The differentiability at $a = 0$ follows because f is squeezed between a "rock and a hard place." For all $x \in \mathbb{R}$, $|f(x)| \leq x^2$. It follows that the difference quotient converges to zero. Indeed, choose $\epsilon > 0$ and let $\delta = \epsilon$. Then for any $x \in \mathbb{R}$, if $0 < |x - 0| < \delta$ then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \frac{|f(x)|}{|x|} \leq \frac{|x|^2}{|x|} = |x| < \delta = \varepsilon.$$

Thus f is differentiable at $a = 0$ since the limit exists: $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$.

4. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which is differentiable on $(0, \infty)$. Suppose that $f(0) = 0$ and $|f'(x)| < M$ for all $x \in (0, \infty)$. Show that for all $x \geq 0$,*

$$|f(x)| \leq M|x|.$$

Proof. If $x = 0$ then $|f(0)| = |0| \leq M|0|$. Thus suppose $x > 0$. Because f is continuous on $[0, x]$ and differentiable on $(0, x)$, my the mean value theorem, there is $c \in (0, x)$ so that

$$|f(x)| = |f(x) - f(0)| = |f'(c)(x - 0)| = |f'(c)|\, |x| \leq M|x|$$

because $|f'(c)| \leq M$ holds for any $c > 0$.

5. *Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable on $\mathbb{R}$ but $f'(x)$ is not continuous on $\mathbb{R}$. Prove that your function has this property.*

Proof. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For all $x \in \mathbb{R}$, this function is squeezed $|f(x)| \leq x^2$. As in problem (3), f is differentiable at zero and $f'(0) = 0$. For $x \neq 0$, the function is the product and composition of differentiable functions, whose derivative is gotten by the product and chain rules

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

For the sequence $x_n = \frac{1}{2\pi n}$ which tends to zero, we have $f'(x_n) = -1$ so that $f'(x_n) \to -1$ as $n \to \infty$. As this is not $f'(0) = 0$, f' is not continuous at 0.

6. *Suppose $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). Suppose $x, y \in (a, b)$ and that m is any number between $f'(x)$ and $f'(y)$. Then there is a z between x and y such that $f'(z) = m$. In other words, the mean value property holds for the derivative, even though the derivative may not be a continuous function.*

Proof. Choose $x, y \in (a, b)$. For convenience, let us assume that $x < y$ and $f'(x) < m < f'(y)$. Other cases are similar. Then the function $g(x) = f(x) - mx$ is continuous on $[x, y]$ and differentiable on (a, b). Also $g'(x) = f'(x) - m < 0$ and $g'(y) = f'(y) - m > 0$.

As in the homework problem, it follows that for $z \in (x, y)$ close enough to x, $g(z) < g(x)$ and for z close enough to y, $g(z) < g(y)$. To see this, let's do the y case. Since g is differentiable at y,

$$g'(y) = \lim_{z \to y-} \frac{g(z) - g(y)}{z - y}$$

so for any $\varepsilon > 0$, there is a $\delta > 0$ so that for any $z \in (x,y)$ so that $y - \delta < z < y$ we have

$$\left| \frac{g(z) - g(y)}{z - y} - g'(y) \right| < \varepsilon.$$

Applying this to $\epsilon = g'(y) > 0$, if $z \in [x,y]$ satisfies $y - \delta < z < y$ then

$$\begin{aligned} g(z) &= g(y) + \left(\frac{g(z) - g(y)}{z - y} - g'(y) \right) (z - y) + g'(y)(z - y) \\ &\le g(y) + \left| \frac{g(z) - g(y)}{z - y} - g'(y) \right| |z - y| + g'(y)(z - y) \\ &< g(y) + g'(y)|z - y| + g'(y)(z - y) = g(y). \end{aligned}$$

Thus it follows that there are points in the interval $z_i \in [x,y]$ such that $g(z_1) < g(x)$ and $g(z_2) < g(y)$. But since g is continuous on $[x,y]$, by the minimum theorem, there is $c \in [x,y]$ so that

$$g(c) = \inf_{z \in [x,y]} g(z).$$

But c cannot be the endpoint because $g(c) \le \min\{g(z_1), g(z_2)\} < \min\{g(x), g(y)\}$, thus $c \in (x,y)$, where g is differentiable. It follows from the theorem about the vanishing of the derivative at a minimum point and the definition of g,

$$f'(c) - m = g'(c) = 0$$

so that at the intermediate point $f'(c) = m$, as desired.

7. *Which is bigger e^π or π^e?*

   Proof. Consider the function $f(x) = e^{-x} x^e$. It is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ since we use logs and exponentials to define $f(x) = \exp(g(x))$ where $g(x) = -x + e \ln x$. Since $g'(x) = -1 + \frac{e}{x}$, it follows that $g'(x) > 0$ for $x \in (0, e)$ and $g'(x) < 0$ for $x \in (e, \infty)$. Hence $g(e) = 0 > g(\pi)$ by the corollary to the Mean Value Theorem relating decreasing to derivatives. Because exp is a strictly increasing function, it is increasing and decreasing on the same intervals as g. It follows from $e < \pi$ that

   $$1 = f(e) = \exp(g(e)) = \exp(0) > \exp(g(\pi)) = f(\pi) = \frac{\pi^e}{e^\pi}$$

   so $\pi^e < e^\pi$.

8. *Suppose that $f : (a, \infty) \to \mathbb{R}$ is differentiable and that $f'(x) \to \infty$ as $x \to \infty$. Then show that f is not uniformly continuous on $(0, \infty)$.*

   Proof. By negating the definition, we are to show f is not uniformly continuous on (a, ∞) which means there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there are $x, y \in (a, \infty)$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \ge \varepsilon$. We show this is true for $\epsilon = 1$. Choose $\delta > 0$. Since $f'(x) \to \infty$, there is $R \in \mathbb{R}$ so that $f'(c) > \frac{2}{\delta}$ whenever $c \in (a, \infty)$ satisfies $c > R$. Now pick an $x \in (a, \infty)$ such that $x > R$. Let $y = x + \frac{\delta}{2}$. We have $y \in (a, \infty)$ and it satisfies $|y - x| = \frac{\delta}{2} < \delta$. Because f is continuous on $[x, y]$ and differentiable on (x, y), by the Mean Value Theorem, there is a $c \in (x, y)$ such that

   $$|f(y) - f(x)| = |f'(c)(y - x)| = f'(c)(y - x) > \frac{2}{\delta} \cdot \frac{\delta}{2} = 1 \ge \varepsilon$$

   because $c > x > R$.

9. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Then f^2 is integrable on $[a,b]$ and*

$$\left(\int_a^b f(x)\,dx\right)^2 \le (b-a)\int_a^b f^2(x)\,dx. \tag{1}$$

Proof. First, for all $x \in [a,b]$, $\inf_{x\in[a,b]} f \le f(x) \le \sup_{x\in[a,b]} f$ so $0 \le f^2(x) \le M$ where $M = \max\{|\inf_{x\in[a,b]} f|^2, |\sup_{x\in[a,b]} f|^2\}$. Thus f is bounded.

Next we show f^2 is integrable using the theorem that says g is integrable om $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ so that $U(g,\mathcal{P}) - L(g,\mathcal{P}) < \varepsilon$. Here the upper and lower sums are

$$U(g,\mathcal{P}) = \sum_{k=1}^n M_k(g)\,(x_k - x_{k-1}), \qquad L(g,\mathcal{P}) = \sum_{k=1}^n m_k(g)\,(x_k - x_{k-1})$$

where

$$M_k(g) = \sup_{x\in[x_{k-1},x_k]} g(x), \qquad m_k(g) = \inf_{x\in[x_{k-1},x_k]} g(x).$$

For any partition, consider three cases for a subinterval: $M_k \le 0$, $m_k \le 0 \le M_k$ and $0 \le m_k$. In the first case for $x \in [x_{k-1}, x_k]$,

$$m_k(f) \le f(x) \le M_k(f) \le 0$$

so that

$$0 \le M_k(f)^2 \le f^2(x) \le m_k(f)^2$$

which implies

$$M_k(f^2) - m_k(f^2) \le m_k(f)^2 - M_k(f)^2 = |m_k(f) + M_k(f)|\,|m_k(f) - M_k(f)| \le 2M(M_k(f) - m_k(f)).$$

In the second case for $x \in [x_{k-1}, x_k]$, $m_k(f) \le 0 \le M_k(f)$. Thus,

$$m_k(f) \le f(x) \le M_k(f)$$

so that

$$0 \le f^2(x) \le \max\{m_k(f)^2, M_k(f)^2\} \le (M_k(f) - m_k(f))^2 \le (|M_k(f)| + |m_k(f)|)(M_k(f) - m_k(f))$$

which implies

$$M_k(f^2) - m_k(f^2) \le M_k(f^2) \le 2M(M_k(f) - m_k(f)).$$

In the third case, for $x \in [x_{k-1}, x_k]$,

$$0 \le m_k(f) \le f(x) \le M_k(f)$$

so that

$$0 \le m_k(f)^2 \le f^2(x) \le M_k(f)^2$$

which implies

$$M_k(f^2) - m_k(f^2) \le M_k(f)^2 - m_k(f)^2 = |M_k(f) + m_k(f)|\,|M_k(f) - m_k(f)| \le 2M(M_k(f) - m_k(f)).$$

Hence, for any subinterval $[x_{k-1}, x_k]$ in every case we have

$$M_k(f^2) - m_k(f^2) \le 2M(M_k(f) - m_k(f)).$$

To prove f^2 is integrable, choose $\varepsilon > 0$. Since f is integrable, there is a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{2M+1}.$$

Then the same partition applied to f^2 yields

$$\begin{aligned}
U(f^2,\mathcal{P}) - L(f^2,\mathcal{P}) &= \sum_{k=1}^{n} (M_k(f^2) - m_k(f^2))\,(x_k - x_{k-1}) \\
&\le \sum_{k=1}^{n} 2M(M_k(f) - m_k(f))\,(x_k - x_{k-1}) \\
&= 2M(U(f,\mathcal{P}) - L(f,\mathcal{P})) \\
&< \frac{2M\varepsilon}{2M+1} < \varepsilon
\end{aligned}$$

Thus f^2 is integrable.

Inequality (1) is known as the Schwartz Inequality. Its proof is a little trick. The inequality is trivial if $a = b$, so we assume $a < b$. For each $t \in \mathbb{R}$ the function $(f(x) + t)^2$ is integrable since it is the square of an integrable function $f(x) + t$. It is nonnegative, thus for all $t \in \mathbb{R}$,

$$0 \le \int_a^b (f(x) + t)^2\,dx = \int_a^b f(x)^2\,dx + 2t\int_a^b f(x)\,dx + t^2\int_a^b dx = \alpha + 2\beta t + \gamma t^2.$$

The quadratic function is minimized when $t = -\frac{\beta}{\gamma}$. Substituting this t,

$$0 \le \alpha - \frac{2\beta^2}{\gamma} + \frac{\gamma\beta^2}{\gamma^2} = \alpha - \frac{\beta^2}{\gamma} = \int_a^b f(x)^2\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)^2$$

which is the Schwartz Inequality (1).

10. *Let $f_n, f : [a,b] \to \mathbb{R}$ be functions defined on a closed, bounded interval. Assume that f_n are bounded and integrable, and that $f_n \to f$ uniformly as $n \to \infty$. Then f is integrable and we can interchange limit and integral*

$$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b f(x)\,dx. \tag{2}$$

Proof. First, we show f is bounded. Since $f_n \to f$ converges uniformly, for every $\varepsilon > 0$, there is an $N(\epsilon) \in \mathbb{R}$ so that if $n \in \mathbb{N}$ satisfies $n > N(\varepsilon)$ and $x \in [a,b]$, then $|f_n(x) - f(x)| < \epsilon$. Taking $\epsilon = 1$, and fix an $n \in \mathbb{N}$ large enough so $n > N(1)$, then any $x \in [a,b]$ satisfies

$$|f(x)| = |f_n(x) + f(x) - f_n(x)| \le |f_n(x)| + |f(x) - f_n(x)| \le \sup_{x\in[a,b]} |f_n(x)| + 1$$

which is finite because f_n is bounded. Thus f is bounded.

Next we show f is integrable. For this purpose, we use the theorem that says f is integrable om $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ so that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$. Here the upper and lower sums are

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} M_k(f)\,(x_k - x_{k-1}), \qquad L(f,\mathcal{P}) = \sum_{k=1}^{n} m_k(f)\,(x_k - x_{k-1})$$

where

$$M_k(f) = \sup_{x \in [x_{k-1}, x_k]} f(x), \qquad m_k(f) = \inf_{x \in [x_{k-1}, x_k]} f(x).$$

Now choose $\varepsilon > 0$. We approximate f by an f_ℓ, then choose a partition that is good for f_ℓ and then show it is good for f. Since the convergence is uniform, there is $N \in R$ so that whenever $\ell \in \mathbb{N}$ satisfies $\ell > N$ and every $x \in [a, b]$ we have

$$|f_\ell(x) - f(x)| < \frac{\varepsilon}{6(b-a)+6}. \tag{3}$$

We pick one such ℓ to show integrable. Thus, for every $x \in [a, b]$,

$$f_\ell(x) - \frac{\varepsilon}{6(b-a)+6} < f(x) < f_\ell(x) + \frac{\varepsilon}{6(b-a)+6}.$$

Now f_ℓ is integrable, so by the theorem, there is a partition of $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ so that $U(f_\ell, \mathcal{P}) - L(f_\ell, \mathcal{P}) < \frac{\varepsilon}{3}$. Hence taking inf and sup over $[x_{k-1}, x_k]$,

$$\begin{aligned}
m_k(f_\ell) - \frac{\varepsilon}{6(b-a)+6} &\le \inf_{x \in [x_{k-1}, x_k]} f_\ell - \frac{\varepsilon}{6(b-a)+6} \le \inf_{x \in [x_{k-1}, x_k]} f = m_k(f), \\
M_k(f) = \sup_{x \in [x_{k-1}, x_k]} f &\le \sup_{x \in [x_{k-1}, x_k]} f_\ell + \frac{\varepsilon}{6(b-a)+6} \le M_k(f_\ell) + \frac{\varepsilon}{6(b-a)+6}.
\end{aligned}$$

It follows that

$$M_k(f) - m_k(f) \le M_k(f_\ell) - m_k(f_\ell) + \frac{\varepsilon}{3(b-a)+3}.$$

Summing over the subintervals,

$$\begin{aligned}
U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{k=1}^{n} \big(M_k(f) - m_k(f)\big)(x_k - x_{k-1}) \\
&\le \sum_{k=1}^{n} \left(M_k(f_\ell) - m_k(f_\ell) + \frac{\varepsilon}{3(b-a)+3}\right)(x_k - x_{k-1}) \\
&= U(f_\ell, \mathcal{P}) - L(f_\ell, \mathcal{P}) + \frac{\varepsilon}{3(b-a)+3} \sum_{k=1}^{n} (x_k - x_{k-1}) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon(b-a)}{3(b-a)+3} < \epsilon.
\end{aligned}$$

Hence f is integrable.

To show that the limit of the integrals is the integral of the limit, choose $\epsilon > 0$ and let $N \in \mathbb{R}$ as above. Applying (3), for every $n > N$ we get

$$\left| \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \right| \le \int_a^b |f(x) - f_n(x)|\, dx \le \int_a^b \frac{\varepsilon\, dx}{6(b-1)+6} = \frac{\varepsilon(b-a)}{6(b-1)+6} < \varepsilon.$$

Thus we have shown (2).

11. *Define* $\log x = \displaystyle\int_1^x \frac{dt}{t}$ *for* $x > 0$ *as usual. If* $x = e^y$ *is the inverse function of* $y = \log x$*, show that* e^y *is differentiable and* $\dfrac{d}{dy} e^y = e^y$.

The differentiability of $F(x) = \log x$ follows from the Fundamental Theorem of Calculus, which says that if f is integrable on $[a, b]$ for any $0 < a < 1 < b < \infty$ then $F(z) = \int_1^z f(t)\, dt$

is uniformly continuous on $[a,b]$ and if f is continuous at $z \in (a,b)$, then F is differentiable at z and $F'(z) = f(z)$. In our case $f(t) = \frac{1}{t}$ so it is continuous, hence integrable on $[a,b]$ and since f is continuous at z, $F'(z) = \frac{1}{z}$.

Since $F'(z) > 0$, it is strictly increasing, and since $F(z)$ is continuous, the inverse function theorem for continuous functions says that $F^{-1} = \exp : [\log a, \log b] \to \mathbb{R}$ is continuous and strictly increasing. We have defined F on $(0,\infty)$ (by taking a small and b large enough). So choose $w \in F((0,\infty))$ and let $F(z) = w$ be the corresponding point inverse to w. By the theorem on derivatives of inverse functions, which says, if F is monotone on $(0,\infty)$ and differentiable at $z \in (0,\infty)$ and $F'(z) = \frac{1}{z} \neq 0$, then the inverse function is differentiable at $w = F(z)$ and

$$\frac{d}{dy}e^y\bigg|_{y=w} = \frac{d}{dy}F^{-1}(y)\bigg|_{y=w} = \frac{1}{F'(z)} = \frac{1}{1/z} = z = F^{-1}(w) = e^w.$$

12. *Does the improper integral* $\displaystyle\int_{-\infty}^{\infty} \frac{dt}{(t^2+t^4)^{\frac{1}{3}}}$ *converge? Why?*

There are four limits: at $-\infty$, $0-$,$0+$ and ∞. Split the integral into four parts

$$I_1 + I_2 + I_3 + I_4 = \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty}$$

Use the comparison theorem for improper integrals. If f, g are integrable on all subintervals and if $|f(t)| \le g(t)$ for all t and if the improper integral $\displaystyle\int_I g(t)\,dt$ converges then the improper integral $\displaystyle\int_I f(t)\,dt$ converges. For the interval I_2 and I_3, we have for $0 < |t| \le 1$,

$$|f(t)| = \frac{1}{(t^2+t^4)^{\frac{1}{3}}} \le \frac{1}{t^{\frac{2}{3}}} = g(t)$$

and the improper integral conveges

$$\int_0^1 g(t)\,dt = \int_0^1 \frac{dt}{t^{\frac{2}{3}}} = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{dt}{t^{\frac{2}{3}}} = \lim_{\varepsilon\to 0+}\Big[3t^{\frac{1}{3}}\Big]_\varepsilon^1 = \lim_{\varepsilon\to 0+}\Big[3 - 3\varepsilon^{\frac{1}{3}}\Big] = 3.$$

Thus the improper integral I_3 exists. Because $g(t)$ is an even function

$$\int_{-1}^0 g(t)\,dt = \int_{-1}^0 \frac{dt}{t^{\frac{2}{3}}} = \lim_{\varepsilon\to 0-}\int_{-1}^{\varepsilon} \frac{dt}{t^{\frac{2}{3}}} = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{dt}{t^{\frac{2}{3}}} = 3$$

from before. Thus the improper integral I_2 exists also.

For the interval I_1 and I_4, we have for $1 \le |t|$,

$$|f(t)| = \frac{1}{(t^2+t^4)^{\frac{1}{3}}} \le \frac{1}{t^{\frac{4}{3}}} = g(t)$$

and the improper integral conveges

$$\int_1^\infty g(t)\,dt = \int_1^\infty \frac{dt}{t^{\frac{4}{3}}} = \lim_{R\to\infty}\int_1^R \frac{dt}{t^{\frac{4}{3}}} = \lim_{R\to\infty}\Big[-3t^{-\frac{1}{3}}\Big]_1^R = \lim_{R\to\infty}\Big[3 - 3R^{-\frac{1}{3}}\Big] = 3.$$

Thus the improper integral I_4 exists. Because $g(t)$ is an even function

$$\int_{-\infty}^{-1} g(t)\,dt = \int_{-\infty}^{-1} \frac{dt}{t^{\frac{4}{3}}} = \lim_{R\to\infty}\int_{-R}^{-1} \frac{dt}{t^{\frac{4}{3}}} = \lim_{R\to\infty}\int_1^R \frac{dt}{t^{\frac{4}{3}}} = 3$$

from before. Thus the improper integral I_1 exists also.

13. *Show that if the limit* $\lim_{n\to\infty} \dfrac{|b_k|}{|a_k|} = L$ *exists and if* $\sum_{k=1}^{\infty} a_k$ *converges absolutely then* $\sum_{k=1}^{\infty} b_k$ *converges absolutely.*

The existence of the limit of nonnegative numbers so $L \geq 0$ shows that the series can be compared. There is an $N \in \mathbb{R}$ so that $a_k \neq 0$ and

$$\frac{|b_k|}{|a_k|} < L + 1$$

whenever $k > N$. Hence, for all $k > N$,

$$|b_k| \leq (L+1)|a_k|.$$

Hence, by the regular comparison test, $\sum_{k=1}^{\infty} |b_k|$ is convergent because $\sum_{k=1}^{\infty}(L+1)|a_k|$ is convergent by assumption.

14. *Determine whether* $\sum_{k=1}^{\infty}(-1)^k \dfrac{(k!)^2}{(2k)!}$ *is absolutely convergent, conditionally convergent or divergent.*

To check absolute convergence, use the ratio test.

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\frac{((k+1)!)^2}{(2k+2)!}}{\frac{(k!)^2}{(2k)!}} = \lim_{k\to\infty} \frac{(k+1)!\cdot(k+1)!}{(2k+2)!}\cdot\frac{(2k)!}{k!\cdot k!} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4}.$$

Since $\rho < 1$, the series is absolutely convergent.

15. *Let* $a^+ = a$ *if* $a \geq 0$ *and* $a^+ = 0$ *if* $a < 0$. *Similarly, let* $a^- = \min\{0, a\}$. *Show that if* $A = \sum_{k=1}^{\infty} a_k$ *is conditionally convergent, then the series*

$$P = \sum_{k=1}^{\infty} a_k^+, \qquad M = \sum_{k=1}^{\infty} a_k^-$$

are both divergent.

Argue by contradiction. We assume that A is conditionally convergent and both P and M are not divergent. Thus we may assume that one of the sums, say P, is convergent. Using the fact that $a_k = a_k^+ + a_k^-$, we have the series of differences from two convergent series is convergent and converges to the difference, so

$$M = \sum_{k=1}^{\infty} a_k^- = \sum_{k=1}^{\infty}(a_k - a_k^+) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} a_k^+ = A - P$$

converges also. Similarly if M converges then so does P. Thus both P and M converge. It follows that A is absolutely convergent. This is because $|a_k| = a_k^+ - a_k^-$. Again, the convergence of the sum of differences follows from the convergence of the individual series

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty}(a_k^+ - a_k^-) = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^- = P - M.$$

Thus we have shown that A is absolutely convergent, hence not conditionally convergent.

16. *Show that if* $A = \sum_{k=1}^{\infty} a_k$ *is convergent and* $\{b_k\}$ *is bounded and monotone, then* $A = \sum_{k=1}^{\infty} a_k b_k$ *is convergent. (This is known as Abel's Test.)*

This one relies on a trick called Abel's Summation by Parts. If $S_k = \sum_{j=1}^{k} a_j$ is the partial sum, then

$$\sum_{k=1}^{n} a_k b_k = S_n b_{n+1} + \sum_{k=1}^{n} S_k (b_k - b_{k+1}). \tag{4}$$

To see this, observe that for $k \in \mathbb{N}$,

$$a_k b_k = (S_k - S_{k-1}) b_k = S_k (b_k - b_{k+1}) + (S_k b_{k+1} - S_{k-1} b_k)$$

where we understand $S_0 = 0$. Now summing gives (4), noting that the second parenthesized term telescopes.

Since $\{b_n\}$ is bounded and monotone, it is convergent: $b_n \to B$ as $n \to \infty$. Since A is convergent, $S_n \to A$ as $n \to \infty$. Thus the first term in the partial sum (4) converges to a limit $S_n b_{n+1} \to AB$ as $n \to \infty$.

The sum $B = \sum_{k=1}^{\infty} (b_k - b_{k-1})$ is convergent because $b_n = \sum_{k=1}^{n} (b_k - b_{k-1}) \to B$ as $n \to \infty$ where we have taken $b_0 = 0$. Since $\{b_n\}$ is monotone, the summands have a fixed sign and the convergence is absolute. Finally, since $S_n \to A$ as $n \to \infty$, it is bounded. This implies that the last sum in (4) converges.

To show $T_n = \sum_{k=1}^{n} S_k (b_k - b_{k+1})$ tends to a limit as $n \to \infty$, suppose the bound is $|S_k| \le M$ for all k. Now we check the Cauchy Criterion. Choose $\varepsilon > 0$. By the convergence of $\{b_n\}$, there is an $N \in \mathbb{R}$ so that $m, \ell > N$ implies $|b_{m+1} - b_{\ell+1}| < \frac{\varepsilon}{M+1}$. So for any $m, \ell > N$ so that $\ell > m$,

$$\begin{aligned} |T_m - T_\ell| &= \left| \sum_{k=m+1}^{\ell} S_k (b_k - b_{k+1}) \right| \le \sum_{k=m+1}^{\ell} |S_k| \, |b_k - b_{k+1}| \\ &\le \sum_{k=m+1}^{\ell} M |b_k - b_{k+1}| = \left| \sum_{k=m+1}^{\ell} M (b_k - b_{k+1}) \right| = M |b_{m+1} - b_{\ell+1}| < \varepsilon. \end{aligned}$$

Thus we have shown $\{T_n\}$ is Cauchy, hence convergent.