Solution to Homework 1.

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Problems 31[6,7,8,9] are from *Introduction to Analysis 3rd ed.* by William R. Wade, Prentice Hall 2004. Problems 425[B1.1, B1.2, B1.2] are from *Introduction to Analysis* by A. Mattuck, Prentice Hall 1999. I have tried to put in all of the details, especially trying to mention the specific hypothesis which is used to deduce the next step in each sentence. These may often be clear from context and could be written more compactly. The footnotes are not essential for the writeup of the problems. They are about heuristics (problem solving) or meta-language (the how and why of proofs.)

(6.) Theorem. *Let X, Y be sets and $f : X \to Y$ be a function. Then the follosing are equivalent:*

(a.) *f is one-to-one.*
(b.) *$f(A - B) = f(A) - f(B)$ for all subsets $A, B \subseteq X$.*
(c.) *$E = f^{-1}(f(E))$ for all subsets $E \subseteq X$.*
(d.) *$f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$.*

Proof. We show that (a.) implies (b.) implies (c.) implies (d.) implies (a.). These inequalities will use some logical equivalences and inequalities about sets and functions that don't use the assumption that f is one-to-one. These are proved separately to try to simplify the presentations. Also we show two other implications (b.) implies (d.) and (c.) implies (a.) which occurred in some students' proofs.

First we assume (a.) f is one-to-one to show that (b.) $f(A - B) = f(A) - f(B)$ for all subsets $A, B \subseteq X$. We shall do this by showing first the set containment statement "$\supseteq$" and then the statement "$\subseteq$."

The containment statement $f(A - B) \supseteq f(A) - f(B)$ does not require the assumption of one-to-one and is done in (e).

The statement $f(A - B) \subseteq f(A) - f(B)$ requires the assumption of one-to-one. Choose $y \in f(A - B)$ to show that $y \in f(A) - f(B)$. $y \in f(A - B)$ means there is an $x \in A - B$ so that $f(x) = y$. Thus $x \in A$ and $x \notin B$. Now $x \in A$ implies $y = f(x) \in f(A)$. We claim that $x \notin B$ imlies that $f(x) = y \notin f(B)$[1]. The claim is equivalent to $y = f(x) \in f(B)$ implies $x \in B$ by contraposition. But $f(x) \in f(B)$ implies that there is $b \in B$ so that $f(x) = f(b)$. Now, since f is one-to-one, it follows that $x = b$. Thus $x = b \in B$ and the claim is verified. Putting these together, $y = f(x) \in f(A)$ and $y = f(x) \notin f(B)$. Thus $y = f(x) \in f(A) - f(B)$ and the statement is proved. Both containment statements imply $f(A - B) = f(A) - f(B)$ and the first part is complete.

Second we assume (b.) $f(A - B) = f(A) - f(B)$ for all subsets $A, B \subseteq X$ to show (c.) $E = f^{-1}(f(E))$ for all $E \subseteq X$. We shall do this by showing first "$\subseteq$" and then "$\supseteq$."

$E \subseteq f^{-1}(f(E))$ holds without any assumptions[2] and is shown in part (g.)

To show $E = f^{-1}(f(E))$, we need to somehow reduce this to a statement about differences. Since we have shown that the set on the right contains the set on the left, we have $f^{-1}(f(E)) = E \cup \left[f^{-1}(f(E)) - E \right]$

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[1]note that this is the *converse* of the claim in (h.)

[2]In fact this assertion is like the "who's buried in Grant's tomb?" type of question. In English, the statement amounts to "E is among the points that take values in the values taken by points of E."

and the equality follows if we can show $f^{-1}(f(E)) - E = \emptyset$. Indeed, by the assumption (b.) and then by part (j.), we have

$$f\big[f^{-1}(f(E)) - E\big] = f\big[f^{-1}(f(E))\big] - f\big[E\big] = f(E) - f(E) = \emptyset.$$

But whenever $f(S) = \emptyset$ for some $S \subseteq X$ we must have $S = \emptyset$, for otherwise $S \neq \emptyset$ implies there is an $x \in S$ which implies $f(x) \in f(S)$ so $f(S) \neq \emptyset$. Applying this to $S = f^{-1}(f(E)) - E$ we get $f^{-1}(f(E)) - E = \emptyset$ and the second part is complete.

Third, we assume (c.) $E = f^{-1}(f(E))$ for all $E \subseteq X$ to show (d.) $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$. We shall do this by showing first the containment "$\subseteq$" and then the containment "$\supseteq$."

$f(A \cap B) \subseteq f(A) \cap f(B)$ follows without any assumption on f and is proved in part (f.)

To show $f(A) \cap f(B) \subseteq f(A \cap B)$ we shall establish that $E = f^{-1}(f(E))$ for all $E \subseteq X$ implies $f(A) \cap f(B) - f(A \cap B) = \emptyset$ for all $A, B \subseteq X$. Hence

$$f(A) \cap f(B) = f(A \cap B) \cup \big[\big(f(A) \cap f(B)\big) - f(A \cap B)\big] = f(A \cap B) \cup \emptyset = f(A \cap B)$$

completes the proof.

The statement is equivalent by contraposition to showing $\big(f(A) \cap f(B)\big) - f(A \cap B) \neq \emptyset$ for some $A, B \subseteq X$ implies $E \neq f^{-1}(f(E))$ for some $E \subseteq X$. Suppose we have such $A, B \subseteq X$ so that $\big(f(A) \cap f(B)\big) - f(A \cap B) \neq \emptyset$. Thus, there is a $y \in \big(f(A) \cap f(B)\big) - f(A \cap B)$. Being in a difference of sets implies $y \in f(A) \cap f(B)$ and $y \notin f(A \cap B)$. Now $y \in f(A) \cap f(B)$ implies $y \in f(A)$ and $y \in f(B)$. But $y \in f(A)$ implies that there is an $a \in A$ so that $f(a) = y$. Similarly, $y \in f(B)$ implies there is a $b \in B$ so that $f(b) = y$. Also $f(a) \notin f(A \cap B)$ implies $a \notin A \cap B$ by part (h.), where this time $A \cap B$ plays the role of C. But $a \in A$ and $a \notin A \cap B$ means that $a \in A - (A \cap B) = A - B$. The last equality is shown in part (i.) In particular $a \notin B$ but $b \in B$ so that $a \neq b$. Now our candidate for the set is $E = \{a\}$. To show that $E \neq f^{-1}(f(E))$, we have $f(E) = f(\{a\}) = \{f(a)\} = \{y\}$. On the one hand, $f(a) = y \in f(E)$ so $E \subseteq f^{-1}(f(E))$. On the other hand $f(b) = y \in f(E)$ so that $b \in f^{-1}(f(E))$. Thus $f^{-1}(f(E))$ is a set of two or more elements since it contains both a and b, $a \neq b$. In particular, $f^{-1}(f(E))$ cannot equal E which is a set with exactly one element, which was to be shown.

Fourth, we assume (d.) $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$ to show (a.) f is one-to-one. Thus we wish to show that when we choose $x_1, x_2 \in X$ so that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Let the sets $A = \{x_1\}$ and $B = \{x_2\}$ so that $f(A) = \{f(x_1)\}$ and $f(B) = \{f(x_2)\}$. Since $x_1 \neq x_2$ we have $A \cap B = \emptyset$. It follows from the assumption (d.) that

$$\emptyset = f(\emptyset) = f(A \cap B) = f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\}.$$

It follows that $f(x_1) \neq f(x_2)$ and we are done.

The proof of Theorem 6 is complete. $\qquad\square$

Fifth we assume (b.) $f(A - B) = f(A) - f(B)$ for all subsets $A, B \subseteq X$ to show (d.) $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$. We shall do this by showing a sequence of equalities of sets. Let $A, B \subseteq X$ be subsets. Thus $A \cap B = A - (A - B)$ is shown in part (k.) Using this equation and the assumption (b.) twice,

$$f(A \cap B) = f\big(A - (A - B)\big) = f(A) - f(A - B) = f(A) - \big(f(A) - f(B)\big) = f(A) \cap f(B),$$

and we are done.

Sixth, we assume (c.) $E = f^{-1}(f(E))$ for all $E \subseteq X$ to show (a.) f is one-to-one. The idea is to use Theorem 1.31: if $f : X \to Z$ is onto and if there is a function $g : Z \to X$ such that $f(g(y)) = y$ for all $y \in Z$ and $g(f(x)) = x$ for all $x \in X$ then f is one-to-one. Every function is onto its image $f : X \to f(X)$, thus we shall let $Z = f(X) \subseteq Y$ in the theorem. The one-to-oneness of the function $f : X \to f(X)$ implies the

one-to-oneness of the function $f : X \to Y$.[3] To construct a function $g : f(X) \to X$ start with choosing any $y \in f(X)$. Hence there is $x \in X$ so that $y = f(x)$. Define $g(y) = x$. Henxe $f(g(y)) = f(x) = y$. Thus we have shown that the set $f^{-1}(\{y\})$ is nonempty (it contains x) and that $g(y)$ picks some element from the set $f^{-1}(\{y\})$. We have already shown that by the construction of g, $f(g(y)) = y$ for any $y \in f(X)$. Suppose $x \in X$. Let $y = f(x)$. Then $g(f(x)) \in f^{-1}(\{f(x)\}) = f^{-1}(f(\{x\})) = \{x\}$ by the assumption (c.) It folows that $g(f(x)) = x$ since it is the only element. Thus the function satisfies both conditions to be the inverse, thus $f : X \to f(X)$ is one-to-one by Theorem 1.31, as to be shown.

(e.) $f(A) - f(B) \subseteq f(A - B)$ for all $A, B \subseteq X$. This holds without the asssumption of one-to-one. Choose $y \in f(A) - f(B)$ to show that $y \in f(A - B)$. Then $y \in f(A)$ and $y \notin f(B)$. But $y \in f(A)$ means there is an $x \in A$ so that $f(x) = y$. Also $y = f(x) \notin f(B)$ implies $x \notin B$ by part (h.) Putting these together, there is an $x \in A$ such that $x \notin B$ so $x \in A - B$. Thus $y = f(x) \in f(A - B)$.

(f.) $f(A \cap B) \subseteq f(A) \cap f(B)$ for all $A, B \subseteq X$. We assume $y \in f(A \cap B)$ to show $y \in f(A) \cap f(B)$. But $y \in f(A \cap B)$ means there is $x \in A \cap B$ so that $y = f(x)$. But then $x \in A$ and $x \in B$. Now $x \in A$ implies $y = f(x) \in f(A)$ and $x \in B$ implies $y = f(x) \in f(B)$ by the meaning of $f(A)$ and $f(B)$. Thus $y \in f(A) \cap f(B)$ as to be shown.

(g.) $E \subseteq f^{-1}(f(E))$ for all $E \subseteq X$. Choose $e \in E$ to show $e \in f^{-1}(f(E))$, which is equivalent to showing $f(e) \in f(E)$. But this is true for $e \in E$ by the meaning of $f(E)$.

(h.) Suppose there is a $z \in X$ and a set $C \subseteq X$. If $f(z) \notin f(C)$ then $z \notin C$. The claim is equivalent to $z \in C$ implies $y = f(z) \in f(C)$ by contraposition. In this form, the claim is evidently true, since $f(C)$ is the set of all images of points in C.

(i.) $A - (A \cap B) = A - B$ for all subsets $A, B \subseteq X$. This is easily seen from a Venn diagram or the logical equivalence P and $(Q$ or $R)$ $\iff$ $(P$ and $Q)$ or $(P$ and $R)$ which can also be seen from Venn Diagram or a truth table. $x \in A - (A \cap B)$ if and only if $x \in A$ and $x \notin A \cap B$ if and only if $x \in A$ and not $(x \in A \cap B)$ if and only if $x \in A$ and not $(x \in A$ and $x \in B)$ if and only if $x \in A$ and $(x \notin A$ or $x \notin B)$ if and only if $(x \in A$ and $x \notin A)$ or $(x \in A$ and $x \in B)$ if and only if (FALSE) or $(x \in A \cap B)$ if and only if $x \in A \cap B$.

(j.) $f[f^{-1}(f(E))] = f(E)$ for all $E \subseteq X$. To see "$\subseteq$," $f[f^{-1}(f(E))] =$
$= \{f(x) \in Y : x \in f^{-1}(f(E))\} = \{f(x) \in Y : x \in X \text{ such that } f(x) \in f(E)\} \subseteq f(E)$. To see "$\supseteq$," if $y \in f(E)$ then there is $e \in E$ so $f(e) = y$. Hence $e \in f^{-1}(f(E))$. Thus $y = f(e) \in f(f^{-1}(f(E)))$, and we are done.

(k.) $A \cap B = A - (A - B)$ for all subsets $A, B \subseteq X$. $x \in A - (A - B)$ if and only if $x \in A$ and $x \notin A - B$ if and only if $x \in A$ and not $(x \in A$ and $x \notin B)$ if and only if $x \in A$ and $(x \notin A$ or $x \in B)$. As in part (i.), for statements P, Q, R we have the equivalence P and $(Q$ or $R)$ $\iff$ $(P$ and $R)$ or $(P$ and $R)$. Thus $x \in A$ and $(x \notin A$ or $x \in B)$ if and only if $(x \in A$ and $x \notin A)$ or $(x \in A$ and $x \in B)$ if and only if (FALSE) or $(x \in A$ and $x \in B)$ if and only if $x \in A$ and $x \in B$ if and only if $x \in A \cap B$.

(7.) Theorem. *Let $E_\alpha \subseteq X$ be subsets for all $\alpha \in A$. Then*

$$(1) \qquad \left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c,$$

$$(2) \qquad \left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c.$$

Proof. We shall show the equality of both sides by showing that the condition for an element to be on one side is logically equivalent to its belonging to the other side.

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[3]This is simply because the range happens to be $f(X)$ in both cases. This is a simple idea that looks complicated when written down: define the restricted function $\hat{f} : X \to f(X)$ in terms of the function $f : X \to Y$ by the formula $\hat{f}(x) := f(x)$. f and $\hat{f}$ are the same function except that we consider them to be functions mapping to different sets. Assume that $\hat{f}$ is one-to-one to show that f is one-to-one. Choose $x_1, x_2 \in X$ such that $x_1 \neq x_2$ to show $f(x_1) \neq f(x_2)$. But $\hat{f}$ one-to-one means whenever $x_1, x_2 \in X$ are chosen such that $x_1 \neq x_2$ then $\hat{f}(x_1) \neq \hat{f}(x_2)$. But $f(x_1) = \hat{f}(x_1) \neq \hat{f}(x_2) = f(x_2)$ thus $f(x_1) \neq f(x_2)$ so f is one-to-one.

To see the equation (1), we have to negate a quantified statement

$$x \in \left(\bigcup_{\alpha \in A} E_\alpha \right)^c \iff x \notin \bigcup_{\alpha \in A} E_\alpha$$

$$\iff \text{not} \left(x \in \bigcup_{\alpha \in A} E_\alpha \right)$$

$$\iff \text{not} \ (\exists \alpha \in A) \, (x \in E_\alpha)$$

$$\iff (\forall \alpha \in A) \, (x \notin E_\alpha)$$

$$\iff (\forall \alpha \in A) \, (x \in E_\alpha^c)$$

$$\iff x \in \bigcap_{\alpha \in A} E_\alpha^c.$$

Equation (2) is similar

$$x \in \left(\bigcap_{\alpha \in A} E_\alpha \right)^c \iff x \notin \bigcap_{\alpha \in A} E_\alpha$$

$$\iff \text{not} \left(x \in \bigcap_{\alpha \in A} E_\alpha \right)$$

$$\iff \text{not} \ (\forall \alpha \in A) \, (x \in E_\alpha)$$

$$\iff (\exists \alpha \in A) \, (x \notin E_\alpha)$$

$$\iff (\exists \alpha \in A) \, (x \in E_\alpha^c)$$

$$\iff x \in \bigcup_{\alpha \in A} E_\alpha^c.$$

(7.) Theorem. *Let X, Y be sets and $f : X \to Y$ be a function. Then*
(iii.) If $E_\alpha \subseteq X$ are subsets for $\alpha \in A$, then

$$(3) \qquad f^{-1} \left(\bigcup_{\alpha \in A} E_\alpha \right) = \bigcup_{\alpha \in A} f^{-1} (E_\alpha),$$

$$(4) \qquad f^{-1} \left(\bigcap_{\alpha \in A} E_\alpha \right) \subseteq \bigcap_{\alpha \in A} f^{-1} (E_\alpha).$$

(iv.) If $B, C \subseteq Y$ then

$$(5) \qquad f^{-1}(C - B) = f^{-1}(C) - f^{-1}(B).$$

(v.) If $S \subseteq f(X)$ then $f \left(f^{-1}(S) \right) = E$. If $E \subseteq X$ then $f^{-1} \left(f(E) \right) \supseteq E$.

Proof. (iii) is similar to Problem 7. To see the equation (3),

$$x \in f^{-1} \left(\bigcup_{\alpha \in A} E_\alpha \right) \iff f(x) \in \bigcup_{\alpha \in A} E_\alpha$$

$$\iff (\exists \alpha \in A) \, (f(x) \in E_\alpha)$$

$$\iff (\exists \alpha \in A) \, \left(x \in f^{-1} (E_\alpha) \right)$$

$$\iff x \in \bigcup_{\alpha \in A} f^{-1} (E_\alpha).$$

Equation (4) is similar. $y \in f\left(\bigcap_{\alpha \in A} E_\alpha\right)$ implies $y = f(x)$ for some $x \in X$ so that

$$x \in \bigcap_{\alpha \in A} (E_\alpha) \iff (\forall \alpha \in A)(x \in E_\alpha).$$

Hence

$$(\forall \alpha \in A)(y = f(x) \in f(E_\alpha)) \iff y \in \bigcap_{\alpha \in A} f(E_\alpha).$$

Finally the second assertion: if $E \subseteq X$ then $E \subseteq f^{-1}(f(E))$ is shown in part (6g.) To see the first assertion (which is similar to (6j.),) if $S \subseteq f(X)$, then $S = f\left(f^{-1}(s)\right)$ we are to show $\subseteq$ and $\supseteq$. To see "$\subseteq$," choose $s \in S \subseteq f(X)$. Then there is $x \in X$ so that $f(x) = s$ Thus $x \in f^{-1}(S)$. Hence $s = f(x) \in f\left(f^{-1}(S)\right)$. To see "$\supseteq$," $f\left(f^{-1}(S)\right) = \{f(x) : x \in f^{-1}(S)\} = \{f(x) : x \in X \text{ such that } f(x) \in S\} \subseteq S$.

(9a.) Theorem. *Suppose X, Y, Z are sets and $f : X \to Y$ and $g : Y \to Z$ are functions. Let $g \circ f : X \to Z$ be defined by $g \circ f(x) = g(f(x))$ for all $x \in X$.*
   (i.) If both f and g are one-to-one then $g \circ f$ is one-to-one.
   (ii.) If both f and g are onto, then $g \circ f$ is onto.

Proof. We assume that f is one to one, namely if for some $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$ then $x_1 = x_2$. Also we assume that g is one to one, namely if for some $y_1, y_2 \in Y$ we have $g(y_1) = g(y_2)$ then $y_1 = y_2$. Now suppose for some $x_1, x_2 \in X$ we have $g \circ f(x_1) = g \circ f(x_2)$ to show $x_1 = x_2$. Put $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $g(y_1) = g(f(x_1)) = g \circ f(x_1) = g \circ f(x_2) = g(f(x_2)) = g(y_2)$. Now g one-to-one implies $f(x_1) = y_1 = y_2 = f(x_1)$. But f is also one-to-one which implies $x_1 = x_2$ which was to be shown. Thus $g \circ f$ is one-to-one.

We assume that f is onto, namely for any $y \in Y$ there is an $x \in X$ so that $y = f(x)$. We also assume that g is onto, namely for any $z \in Z$ there is a $y \in Y$ so that $z = g(x)$. Now suppose we choose any $z \in Z$ to show there is an $x \in X$ so that $z = g \circ f(x)$. But for $z \in Z$, g surjective implies there is a $y \in Y$ so that $z = g(y)$. f surjective implies there is $x \in X$ so that $y = f(x)$. Thus $g \circ f(x) = g(f(x)) = g(y) = z$ as to be shown. Thus $g \circ f$ is onto. $\qquad\square$

B.1.1 Exercise. *A tail of a sequence $\{a_n\}_{n \in \mathbb{N}}$ is the sequence after removing the first $N - 1$ terms of $\{a_n\}$. Write the given sentence with quantifiers in the right order; Interchange the order of the first two quantifiers if possible and interpret.*

$$\text{Every tail of } \{a_n\}_{n \in \mathbb{N}} \text{ has a maximal element.}$$

Solution. Recall that the best way to think of a sequence is as a function $a : \mathbb{N} \to \mathbf{R}$ given by $j \mapsto a_j$. The N-th tail is the subsequence $\{a_N, a_{N+1}, a_{N+2}, \dots\}$. The statement can be rendered as

$$(\forall N \in \mathbb{N})(\exists j \in \mathbb{N} : j \geq N)(\forall k \in \mathbb{N} : k \geq N)(a_j \geq a_k)$$

In this form, the first two quantifiers cannot be reversed because the second one depends on the first variable.

B.2.1a. Exercise. *Render in English a statement equivalent of the negated sentence. Then write the given sentence with quantifiers and negations in the right order. Use the rules for negating quantifiers to rewrite the original sentence.*

$$\mathcal{S} := \text{"The sequence } \{a_n\}_{n \in \mathbb{N}} \text{ has no minimum."}$$

Solution. The meaning is that it is not the case that the sequence has a minimum. People usually can intuitively put this in the form "for every term of the sequence there is another term which is smaller." The statement can be rendered as

$$\sim \mathcal{S} \iff \sim (\exists j \in \mathbb{N})(\forall k \in \mathbb{N})(a_j \leq a_k).$$

Then the rules of negation make the equivalent sentence

$$\sim \mathcal{S} \iff (\forall j \in \mathbb{N})(\exists k \in \mathbb{N})(a_j > a_k).$$

B.3.1. Exercise. *Write the given sentence with quantifiers in the right order. Use the rules for negating quantifiers to negate the sentence. Then verify that the given is a counterexample to the sentence.*

$$f \text{ is bounded on the interval } I = (0,1]. \quad f(x) = \frac{1}{x} \text{ is not bounded above on } I.$$

Solution. The statement can be rendered as

$$(\exists B \in \mathbb{R}) \, (\forall x \in I) \, (f(x) \le B).$$

Then the negated sentence is

$$\sim (\exists B \in \mathbb{R}) \, (\forall x \in I) \, (f(x) \le B) \iff (\forall B \in \mathbb{R}) \, (\exists x \in I) \, (f(x) > B).$$

To show that the negated sentence is true for the function $f(x) = \frac{1}{x}$ one has the demonstrate that no matter what real number B is given, one can find an $x \in I$ so that $f(x) > B$.[4]

To show that for every $B \in \mathbb{R}$, there is an $x \in I$ such that $f(x) > B$, suppose that $B \in R$ is given. Let

$$x = \begin{cases} \frac{1}{2B}, & \text{if } B > 1; \\ \frac{1}{2}, & \text{if } B \le 1. \end{cases}$$

First we show $x \in I$. In case $B > 1$ then $0 < x = \frac{1}{2B} < 1$ so $x \in I$. In the other case $B \le 1$, then $x = \frac{1}{2}$ which also satisfies $x \in I$.

Second we show that $f(x) > B$. In case $B > 1$, $x = \frac{1}{2B}$ and $f(x) = f\left(\frac{1}{2B}\right) = 2B > B$. However if $B \le 1$ is the case, $x = \frac{1}{2}$ and $f(x) = f\left(\frac{1}{2}\right) = 2 > 1 \ge B$. In both cases $f(x) > B$ as was to be shown. Thus $f(x) = \frac{1}{x}$ is not bounded on I.[5] □

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[4]One could try using the fact that $f(x)$ blows up as $x \to 0$, but this is complicated since it involves three quantifiers, as we shall see in a couple of weeks. It is always better to give a simple direct argument. For existence statements, there is no better argument than to provide x by a formula, and then verify that this x has the desired properties.

[5]Of course the written form of the proof is not the way it was discovered. One starts by trying to solve for x that satisfies $f(x) = B$ first (to get $x = 1/B$,) and then trying to modify such an x so that $f(x) > B$ (say by trying $x = (2B)^{-1}$.) Then, noting that some concession has to be made for small and negative values of B since for such $x \notin I$, motivates splitting the answer into two cases.