Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs Sept. 24, 2009

From exams given Sept. 20 and Oct. 24, 2004.

(1.) *Using only the definition of convergence of a sequence, show* $\lim_{k\to\infty} x_k = 1$ *where* $x_k = \dfrac{k^2-4k}{k^2-8}$ *for all* $k \in \mathbf{N}$.

Proof. Choose $\varepsilon > 0$. By the Archimedean Principle, there is an $N \in \mathbf{N}$ so that $N > \max\left\{3, \dfrac{12}{\varepsilon}\right\}$. For any choice of $k \in \mathbf{N}$ such that $k \ge N$ we have $k \ge 4$ so $8 \le 2k$ and $8 \le \frac{1}{2}k^2$ so that

$$|x_k - 1| = \left|\frac{k^2-4k}{k^2-8} - 1\right| = \left|\frac{(k^2-4k)-(k^2-8)}{k^2-8}\right| = \frac{|8-4k|}{|k^2-8|} \le \frac{|8|+|4k|}{|k^2|-|8|} \le \frac{2k+4k}{k^2-\frac{1}{2}k^2} = \frac{12}{k} \le \frac{12}{N} < \varepsilon. \quad \square$$

(2.) *Suppose that* $x_n \to 0$ *as* $n \to \infty$ *and that the sequence* $\{y_n\}_{n\in\mathbf{N}}$ *is bounded. Show that* $x_n \cdot y_n \to 0$ *as* $n \to \infty$.

Proof. We show that $x_n \cdot y_n \to 0$, which is equivalent to: for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$ such that $k \ge N$ we have $|x_k \cdot y_k - 0| < \varepsilon$.

Since we are given that $\{y_n\}_{n\in\mathbf{N}}$ is bounded, there is a number $C \in \mathbf{R}$ so that for all $k \in \mathbf{N}$ we have $|y_k| < C$.

Choose $\varepsilon > 0$. Since we are given that $x_n \to 0$ as $n \to \infty$, there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$ we have $|x_k| \le \dfrac{\varepsilon}{C}$. For any choice of $k \in \mathbf{N}$ so that $k \ge N$ we have

$$|x_k \cdot y_k - 0| = |x_k - 0| \cdot |y_k| < \frac{\varepsilon}{C} \cdot C = \varepsilon. \quad \square$$

(3.) *Let* $E \subseteq \mathbf{R}$ *be given by* $E = \left\{2 - \frac{1}{n^3} : n \in \mathbf{N}\right\}$. *Find* $s = \sup E$. *Prove that* s *is the supremum (least upper bound) for the set* E.

$s = 2$. To show $s = \sup E$ we must show that it is an upper bound and that it is the least upper bound.

To show that it is an upper bound, choose $x \in E$. Hence $x = 2 - \frac{1}{n^3}$ for some $n \in \mathbf{N}$. But as $n > 0$ we have $n^{-3} > 0$ so $x = 2 - n^{-3} < 2$. This is because $n^{-1} > 0$ implies $n^{-3} = (n^{-1})^3 > 0$ so $-n^{-3} < 0$. Adding 2 to both sides, $x = 2 - n^{-3} < 2 + 0 = 2$. Thus every $x \in E$ has $x \le 2$, that is, 2 is an upper bound for E.

To show that it is the least upper bound, we have to show that for all $\varepsilon > 0$ there is an $x \in E$ so that $s - \varepsilon < x$. Thus choose $\varepsilon > 0$. As $1 > 0$ and $1/\varepsilon > 0$, by the Archimedean Principle, there is an $n \in \mathbf{N}$ so that $n \cdot 1 > 1/\varepsilon$. n^3 is even larger, as can be seen by multiplying $n \ge 1$ by $n > 0$ and $n^2 > 0$ to get $n^3 \ge n^2$ and $n^2 \ge n$ so that $n^3 \ge n^2 \ge n > 1/\varepsilon$. Thus $n^{-3} < \varepsilon$ so $-n^{-3} > -\varepsilon$. Adding 2 to both sides, $2 - \varepsilon < 2 - n^{-3} = x$. As this is the form of numbers in E, we have found an $x \in E$ so that $s - \varepsilon < x$. Thus s is the least upper bound. The argument is complete.

(4.) *Assuming only the field axioms for* $\mathbf{R}$ *(Postulate 1, on pages 2–3 of the text,) deduce that for every* $a, b \in \mathbf{R}$, $-(a+b) = (-a) + (-b)$.

We shall show that $u = (-a) + (-b)$ satisfies $(a+b) + u = 0$. By the Existence of Additive Inverse Axiom for $a+b$, there is some $-(a+b)$ such that $(a+b) + (-(a+b)) = 0$. By the uniqueness asserted in the same axiom, as u is also an additive inverse of $(a+b)$ we must have $u = -(a+b)$ proving the assertion. We have

$$\begin{aligned}
(a+b) + ((-a)+(-b)) &= (b+a) + ((-a)+(-b)) && \text{Commutativity of Addition}\\
&= ((b+a)+(-a)) + (-b) && \text{Associativity of Addition}\\
&= (b+(a+(-a))) + (-b) && \text{Associativity of Addition}\\
&= (b+0) + (-b) && \text{Property of Additive Inverse}\\
&= b + (-b) && \text{Property of Additive Identity}\\
&= 0. && \text{Property of Additive Inverse.}
\end{aligned}$$

(5.) *Determine whether the statement is true or false and prove your answer.* Statement: *For all real functions $f : \mathbf{R} \to \mathbf{R}$ and for all pairs of subsets $A, B \subseteq \mathbf{R}$, if $f(A) \subseteq f(B)$ then $A \subseteq B$.*

The statement is false. We establish the negation: There is a function $f : \mathbf{R} \to \mathbf{R}$ and there are sets $A, B \subseteq \mathbf{R}$ such that $f(A) \subseteq f(B)$ but not $A \subseteq B$.

Let $f(x) = x^2$. (Any function that is not one-to-one will do!) Let $A = \{1, -1\}$ and $B = \{1\}$. We have $f(A) = f(\{1, -1\}) = \{1\}$ and $f(B) = f(\{1\}) = \{1\}$ so that $f(A) \subseteq f(B)$ since they are equal, but A is not contained in B since A has two elements whereas B has one.

(6.) *Determine whether the statements are true or false. If the statement is true, give the reason. If the statement is false, provide a counterexample.*

(a.) *If $x_n \to a$ as $n \to \infty$ then $(x_{n+1} - x_n) \to 0$ as $n \to \infty$.*

TRUE. By the difference theorem for limits, $\lim\limits_{n\to\infty} (x_{n+1} - x_n) = \left(\lim\limits_{n\to\infty} x_{n+1}\right) - \left(\lim\limits_{n\to\infty} x_n\right) = a - a = 0.$

(b.) *If $\{x_n\}_{n\in\mathbf{N}}$ is a bounded sequence then it converges.*

FALSE. $x_n = (-1)^n$ is bounded ($|x_n| \le 1$ for all $n \in \mathbf{N}$) but it does not converge.

(c.) *If $\{xn\}_{n \in \mathbf{N}}$ has a subsequence $\{xn_j\}_{j \in \mathbf{N}}$ that diverges to infinity, $\lim\limits_{j\to\infty} xn_j = \infty$, then the sequence itself diverges to inifty: $\lim\limits_{n\to\infty} x_n = \infty$.*

FALSE. $y_n = (-1)^n n$ has a subsequence $x_{2j} = (-1)^{2j}(2j) = 2j \to \infty$ as $j \to \infty$. But it also has a subsequence $x_{2j+1} = (-1)^{2j+1}(2j+1) = -2j - 1 \to -\infty$ as $j \to \infty$, so that y_n does not diverge to positive infinity. (If it did, *every* subsequence would have to diverge to positive infinity also.)

More Practice Problems.

(E1.) *Using only the definition of convergence, prove that the sequence $\{x_n\}_{n\in\mathbf{N}}$ converges, where $x_n = \dfrac{n + (-1)^n}{n+4}$. [Hint: find the limit first.]*

Proof. To show $x_n \to 1$ as $n \to \infty$, or equivalently, for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$ we have $|x_k - 1| < \varepsilon$.

Choose $\varepsilon > 0$. By the Archimidean Principle, there is an $N \in \mathbf{N}$ so that $N > 5/\varepsilon$. Choose $k \ge N$. Then

$$\begin{aligned} |x_k - 1| &= \left| \frac{k + (-1)^k}{k+4} - 1 \right| = \left| \frac{k + (-1)^k - (k+4)}{k+4} \right| = \left| \frac{(-1)^k + (-4)}{k+4} \right| \\ &= \frac{|(-1)^k + (-4)|}{|k+4|} \le \frac{|(-1)^k| + |-4|}{|k|} = \frac{5}{k} \le \frac{5}{N} < \varepsilon. \quad \square \end{aligned}$$

(E2.) *Suppose $\{x_n\}_{\in\mathbf{N}}$ is a real sequence such that $x_n \to a$ as $n \to \infty$. Using only the definition of convergence, show that the sequence of squares converges and $\lim_{n\to\infty}\left(x_n{}^2\right) = a^2$.*

Proof. To show for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$, $|x_k^2 - a^2| < \varepsilon$.

We are assuming $x_n \to a$ as $n \to \infty$, which means, for all $\varepsilon_1 > 0$ there is an $N_1 \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $j \ge N_1$, $|x_j - a| < \varepsilon$. For $\varepsilon_1 = 1$, there is $N_2 \in \mathbf{N}$ so that for all $j \ge N_2$, $|x_j - 1| < 1$. Hence, for all $j \ge N_1$, $|x_j + a| = |x_j - a + 2a| \le |x_j - a| + |2a| < 1 + 2|a|$. Now choose $\varepsilon > 0$. As $x_n \to a$, there is an $N_3 \in \mathbf{N}$ so that for all $\ell \ge N_3$, $|x_j - a| < \varepsilon/(1 + 2|a|)$. Let $N = \max\{N_2, N_3\}$. For any choice of $k \ge N$, we have

$$|x_k^2 - a^2| = |(x_k - a)(x_k + a)| = |x_k - a| \cdot |x_k + a| < \frac{\varepsilon}{1 + 2|a|} \cdot (1 + 2|a|) = \varepsilon. \quad \square$$

(E3.) *Assume that x_n, y_n, z_n are real sequences such that $x_n \to 0$ as $n \to \infty$, $\{y_n\}_{n\in\mathbf{N}}$ is bounded and $z_n \to \infty$ as $n \to \infty$. For each part, determine whether the statement is TRUE or FALSE. If the statement is true, give a justification. If the statement is false, give a counterexample. You may use theorems about sequences.*

(a.) $\{x_n + y_n\}_{n\in\mathbf{N}}$ has a convergent subsequence.

TRUE: As $\{x_n\}_{n\in\mathbf{N}}$ converges, it must be bounded by Theorem 2.8. This means that there is an $M_1 \in \mathbf{R}$ so that for all $k \in \mathbf{N}$, $|x_k| \le M_1$. We are given by hypothesis that $\{y_n\}_{n\in\mathbf{N}}$ is bounded. This means that there is an $M_2 \in \mathbf{R}$ so that for all $k \in \mathbf{N}$, $|y_k| \le M_2$. Adding, we find for all $k \in \mathbf{N}$, $|x_k + y_k| \le |x_k| + |y_k| \le M_1 + M_2$,

thus the sequence $\{(x_n + y_n)\}_{n\in\mathbf{N}}$ is bounded. Thus by the Bolzano-Weierstraß Theorem, the sum sequence has a convergent subsequence.
(b.) $\{x_n z_n\}_{n\in\mathbf{N}}$ *is bounded.*
FALSE: Take $x_n = n^{-1}$. Then $x_m \to 0$ as $n \to \infty$. Take $z_n = n^2$. Then $z_n \to \infty$ and $\{z_n\}_{n\in\mathbf{N}}$ diverges to infinity. Finally $x_n z_n = n \to \infty$ as $n \to \infty$ so it is not bounded.
(c.) $y_n z_n \to \infty$ *as* $n \to \infty$.
FALSE: Take $y_n = n^{-1}$. Thus $y_n \to 0$ as $n \to \infty$ so it is bounded by Theorem 2.8. Take $z_n = n$ so $z_n \to \infty$ as $n \to \infty$ so z_n is not bounded. However $y_n z_n = 1$ for all $n \in \mathbf{N}$, so $|y_n z_n| \le 1$ for all n so $\{y_n z_n\}_{n\in\mathbf{N}}$ is bounded so does not diverge to infinity.

(E4.) *Suppose* $0 \le x_1 \le 2$ *and define* $x_{n+1} = \sqrt{2 + x_n}$ *for* $n \in \mathbf{N}$.
(a.) Using induction, show that x_n *is monotone increasing and bounded above.*
Proof. We claim that whenever $0 < a < 2$ then $0 < a < \sqrt{2+a} < 2$. The first inequality is given by the hypothesis that $0 < a$. The third inequality follows from the hypothesis $0 < a < 2$ because it implies $0 < a + 2 < 2 + 2$ so that $\sqrt{2+a} < \sqrt{2+2} = 2$. We have used $0 \le a < b$ implies $\sqrt{a} < \sqrt{b}$. The middle inequality follows from $0 < a < 2$ becase this implies $a - 2 < 0$ and $a + 1 > 0$. Thus $a^2 - a - 2 = (a-2)(a+1) < 0$. Using also that $a^2 \ge 0$ we get $0 \le a^2 < a + 2$. Finally we conclude from taking square roots that $a < \sqrt{a+2}$. The claim is verified.
Now for arbitrary $0 < x_1 < 2$ we define inductively $x_{n+1} = \sqrt{2 + x_n}$. Finally we prove $0 < n_n < x_{n+1} < 2$ for all $n \in \mathbf{N}$. We argue by induction. For the base case $n = 1$, we apply the first claim with $a = x_1$ which satisfies $0 < x_1 < 2$ to conclude $0 < x_1 < x_2 = \sqrt{x_1 + 2} < 2$. For the induction step, we assume that $0 < x_n < x_{n+1} < 2$. This implies $0 < x_{n+1} < 2$, so that if we apply the first claim with $a = x_{n+1}$ we conclude $0 < x_{n+1} < x_{n+2} = \sqrt{x_{n+1} + 2} < 2$. The induction proof is complete.

(E5.) *Using only the definition of convergence, show that the sequence* $\{x_n\}_{n\in\mathbf{N}}$ *converges where* $x_n = \frac{n}{3n-1}$.
Proof. We show that $x_n \to \frac{1}{3}$ wnen $n \to \infty$, or equivalently, for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$, $\left|x_k - \frac{1}{3}\right| < \varepsilon$.
Choose $\varepsilon > 0$. By the Archimidean Axiom, there is an $N \in \mathbf{N}$ so that $N > \frac{1}{9\varepsilon} + \frac{1}{3}$. Now for any choice of $k \ge N$, we have $3k - 1 \ge 3N - 1 > \frac{1}{3\varepsilon}$ so that

$$\left|x_k - \frac{1}{3}\right| = \left|\frac{k}{3k-1} - \frac{1}{3}\right| = \left|\frac{3k - (3k-1)}{3(3k-1)}\right| = \frac{1}{3(3k-1)} < \frac{3\varepsilon}{3} = \varepsilon. \quad \square$$

(E6.) *Using only the definition of convergence, show that the sequence* $\{y_n\}_{n\in\mathbf{N}}$ *does not converge, where* $y_n = \frac{1}{n} + (-1)^n$.
Proof. We show that $\{y_n\}_{n\in\mathbf{N}}$ does not converge to any L, or equivalently, for all $L \in \mathbf{R}$ there is an $\varepsilon > 0$ so that for all $N \in \mathbf{N}$ there is a $k \in \mathbf{N}$ such that $k \ge N$ and $|y_k - L| \ge \varepsilon$.
Choose $L \in \mathbf{R}$. Let $\varepsilon = 0.1$. For any choice of $N \in \mathbf{N}$, by the Archimidean Principle, there is a $m \in \mathbf{N}$ so that $2m > N$. If $L \le 0.5$ then let $k = 2m > N$. Then since $\frac{1}{2m} + 1 - L > 0$,

$$|y_k - L| = \left|\frac{1}{k} + (-1)^k - L\right| = \left|\frac{1}{2m} + (-1)^{2m} - L\right| = \left|\frac{1}{2m} + 1 - L\right| = \frac{1}{2m} + 1 - L \ge 0 + 1 - 0.5 > 0.1.$$

If $L > 0.5$ then let $k = 2m + 1 > N$. Then since $\frac{1}{2m+1} - 1 - L < \frac{1}{3} - 1 - 0.5 < 0$,

$$\begin{aligned} |y_k - L| &= \left|\frac{1}{k} + (-1)^k - L\right| = \left|\frac{1}{2m+1} + (-1)^{2m+1} - L\right| \\ &= \left|\frac{1}{2m+1} - 1 - L\right| = -\frac{1}{2m+1} + 1 + L \ge -\frac{1}{3} + 1 + 0.5 > 0.1. \end{aligned}$$

(E7.) *Show using only the definiton of convergence that if* $a_n \to a$ *and* $b_n \to b$ *as* $n \to \infty$ *and* $b \ne 0$ *and* $b_n \ne 0$ *for all* $n \in \mathbf{N}$ *then the quotients converge and the limit of the quotients is the quotient of the limits.*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Proof. We show that $a_n/b_n \to a/b$ when $n \to \infty$, or equivalently, for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$, $|a_k/b_k - a/b| < \varepsilon$.

Since $b_n \to b$ as $n \to \infty$, there is an $N_1 \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N_1$ we have $|b_k - b| < \frac{1}{2}|b|$. Thus, using the reverse triangle inequality, $|b_k| = |b - (b - b_k)| \ge |b| - |b - b_k| > |b| - \frac{1}{2}|b| = \frac{1}{2}|b| > 0$ for all $k \ge N_1$.

Choose $\varepsilon > 0$. Since $a_n \to a$ as $n \to \infty$, there is an $N_2 \in \mathbf{N}$ so that for all $k \ge N_2$, $|a_k - a| < \frac{1}{4}|b|\varepsilon$. Since $b_n \to b$ as $n \to \infty$, there is an $N_3 \in \mathbf{N}$ so that for all $k \ge N_3$, $|b_k - b| < \dfrac{|b|^2\varepsilon}{4|a|+1}$.

Now let $N = \max\{N_1, N_2, N_3\}$. For any choice of $k \in \mathbf{N}$ such that $k \ge N$, we have by the triangle inequlaity and the lower bound on $|b_k|$,

$$\left|\frac{a_k}{b_k} - \frac{a}{b}\right| = \left|\frac{ba_k - ab_k}{bb_k}\right| = \frac{|b(a_k - a) - a(b_k - b)|}{|b|\,|b_k|} \le \frac{|b|\,|a_k - a| + |a|\,|b_k - b|}{\frac{1}{2}|b|^2}$$

$$= \frac{2}{|b|} \cdot |a_k - a| + \frac{2|a|}{|b|^2} \cdot |b_k - b| < \frac{2}{|b|} \cdot \frac{|b|\varepsilon}{4} + \frac{2|a|}{|b|^2} \cdot \frac{|b|^2\varepsilon}{4|a|+1} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

(E8.) *Show that if $x \in \mathbf{R}$ then there is a* <u>*monotone decreasing*</u> *sequence of rationals $q_n \in \mathbf{Q}$ so that $q_n \to x$ as $n \to \infty$.*

Proof. Use the Density of Rationals and the Squeeze Theorem. For each $n \in \mathbf{N}$, let $x_n = x + \frac{1}{n}$. Note that $x_{n+1} = x + \frac{1}{n+1} < x + \frac{1}{n} = x_n$ for all n so that $\{x_n\}_{n\in\mathbf{N}}$ is a decreasing, though not necessarily a rational sequence. By the Density of Rationals Theorem 1.24, there is a rational number $r_n \in \mathbf{Q}$ so that $x_{n+1} < r_n < x_n$ for each $n \in \mathbf{N}$. As both the upper and lower sequences converge to the same limit, $\lim_{n\to\infty} x_{n+1} = x = \lim_{n\to\infty} x_n$, and the sequence $\{r_n\}_{n\in\mathbf{N}}$ is sqeezed in between, by the Squeeze Theorem 2.9, the middle sequence converges also to the same limit $x = \lim_{n\to\infty} r_n$. Finally, as the r_n's are chosen to lie between consecutive terms of a decreasing sequence, the r_n's strictly decrease also: $r_{n+1} < x + \frac{1}{n+1} = x_{n+1} < r_n$ for all $n \in \mathbf{N}$. $\square$

(E9.) *Determine if true or false. If true, give the proof. If false, give a counterexample. Suppose $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ are real sequences such that $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$. Then x_n/y_n converges as $n \to \infty$.*

The statement is FALSE. Let $x_n = n^2$ and $y_n = n$ for all $n \in \mathbf{N}$. Then both $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$. However, $x_n/y_n = n^2/n = n$ which tends to ∞ as $n \to \infty$ so does not converge.

(E10.) *Determine whether the following sequence converges. Justify your answer.*

$$\frac{1}{2},\ \frac{1\cdot 3}{2\cdot 4},\ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\ \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8},\ \ldots$$

The sequence CONVERGES because it is monotone decreasing and bounded below. The sequence may be defined by the recursion $x_1 = \frac{1}{2}$ and $x_{n+1} = \dfrac{2n+1}{2n+2}x_n$ for all $n \in \mathbf{N}$. An induction argument will show that $x_n > 0$ (because each successive term is a positive multiple of the previous which was positive) and $x_{n+1} < x_n$ (because each successive term is a fraction of the previous positive term) for all $n \in \mathbf{N}$. Thus $\{x_n\}_{n\in\mathbf{N}}$ is bounded below (by 0) and decreasing. Be the Monotone Sequence Theorem 2.19, this implies that $\{x_n\}_{n\in\mathbf{N}}$ converges. $\square$

(E11.) *Suppose $\{x_n\}_{n\in\mathbf{N}}$ is a real sequence that has one subsequence $\{x_{n_j}\}_{j\in\mathbf{N}}$ which converges $x_{n_j} \to a$ as $j \to \infty$ and another subsequence $\{x_{m_\ell}\}_{\ell\in\mathbf{N}}$ which converges $x_{m_\ell} \to b$ as $\ell \to \infty$ where a, b are finite real numbers. Show that if $a \ne b$ then the original sequence $\{x_n\}_{n\in\mathbf{N}}$ does not converge, but if $a = b$ then the sequence may or may not be convergent. Give illustrative examples.*

Proof. Assume that $a > b$ and that the subsequences are $x_{n_j} \to a$ as $j \to \infty$ and $x_{m_\ell} \to b$ as $\ell \to \infty$. We show that $\{x_n\}_{n\in\mathbf{N}}$ does not converge to any L, or equivalently, for all $L \in \mathbf{R}$ there is an $\varepsilon > 0$ so that for all $N \in \mathbf{N}$ there is a $k \in \mathbf{N}$ such that $k \ge N$ and $|x_k - L| \ge \varepsilon$.

Choose $L \in \mathbf{R}$. Let $\varepsilon = \frac{1}{8}(a-b)$. As $x_{n_j} \to a$, there is an $N_1 \in \mathbf{N}$ so that for all $j \ge N_1$ we have $|x_{n_j} - a| < \frac{1}{8}(a-b)$. As $x_{m_\ell} \to b$, there is an $N_2 \in \mathbf{N}$ so that for all $\ell \ge N_2$ we have $|x_{m_\ell} - b| < \frac{1}{8}(a-b)$. For any choice of $N \in \mathbf{N}$, by the Archimidean Principle, there is a $j \in \mathbf{N}$ so that $j > \max\{N, N_1, N_2\}$ thus $n_j > \max\{N, N_1, N_2\}$ and $m_j > \max\{N, N_1, N_2\}$. If $L \le \frac{1}{2}(a+b)$ then let $k = n_j$. Then using the reverse triangle inequality and since $a - L > 0$,

$$|x_k - L| = \left|x_{n_j} - L\right| = \left|(a-L) + (x_{n_j} - a)\right| \ge |a - L| - |x_{n_j} - a| > a - \frac{a+b}{2} - \frac{1}{8}(a-b) = \frac{3}{8}(a-b) > \varepsilon.$$

If $L > \frac{1}{2}(a+b)$ then let $k = m_j$. Then using the reverse triangle inequality and since $b - L < 0$,

$$|x_k - L| = \left|x_{m_j} - L\right| = \left|(b-L) + (x_{n_j} - b)\right| \ge |b - L| - |x_{n_j} - b| > \frac{a+b}{2} - b - \frac{1}{8}(a-b) = \frac{3}{8}(a-b) > \varepsilon.$$

Thus no matter what L or N may be, there is an element $k \ge N$ so that $|x_k - L| > \varepsilon$, so the sequence does not converge. $\square$

Consider the sequence $x_n = (-1)^n$ for all n. If $n_j = 4j$ then $x_{n_j} = (-1)^{4j} = 1 \to 1$ as $j \to \infty$. Another subsequence is given by $m_\ell = 4\ell + 2$. Then $x_{m_\ell} = (-1)^{4\ell+2} = 1 \to 1$ as $\ell \to \infty$. This example does not converge, but contains two subsequences that converge to the same limits ($a = b = 1$.) Or we could have taken $m_\ell = 4\ell + 3$. Then $x_{m_\ell} = (-1)^{4\ell+3} = -1 \to -1$ as $\ell \to \infty$. The same non-convergent sequence has another subsequence that converges to a different number, $b = -1$.

On the other hand, if we choose any convergent sequence (e.g. $\xi_n = \frac{1}{n}$), then by Rremark 2.6, any of its subsequences converges to the same limit as the sequence. Using the same indices as before,

$$\xi_{n_j} = \frac{1}{n_j} = \frac{1}{4j} \to 0 \qquad \text{as } j \to 0$$

and

$$\xi_{m_\ell} = \frac{1}{m_\ell} = \frac{1}{4j+3} \to 0 \qquad \text{as } j \to 0.$$

(E12.)*Suppose that the real sequence* $\{x_n\}_{n\in\mathbf{N}}$ *is bounded and that the real sequence* $\{y_n\}_{n\in\mathbf{N}}$ *tends to infinity* $y_n \to \infty$ *as* $n \to \infty$. *Show*

$$\lim_{n\to\infty} (x_n + y_n) = \infty, \qquad [\textit{i.e.} \quad x + \infty = \infty.]$$

Proof. We show that $z_n = x_n + y_n \to \infty$ as $n \to \infty$ which means for all $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$ such that $k \ge N$ we have $z_k > M$.

As $\{x_n\}_{n\in\mathbf{N}}$ is a bounded sequence, there is a $C \in \mathbf{R}$ so that $|x_k| \le C$ for all $k \in \mathbf{N}$. Choose $M \in \mathbf{R}$. As $\{y_n\}_{n\in\mathbf{N}}$ diverges to infinity as $n \to \infty$, there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$ such that $k \ge N$ we have $y_k > M + C$. We show that this N proves the claim for $\{z_n\}_{n\in\mathbf{N}}$. Thus if we choose $k \in \mathbf{N}$ such that $k \ge N$ then

$$z_k = y_k + x_k > (M + C) - |x_k| \ge (M + C) - C = M. \quad \square$$