Math 3210 § 3. **First Midterm Exam** Name: Solutions
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1. *Assume $a \neq 0$. Prove that for every $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}. \tag{1}$$

We prove the formula by induction.

BASE CASE. For $n = 1$, the left side of formula (1) is the first term $\displaystyle\sum_{k=1}^{1} \frac{a-1}{a^k} = \frac{a-1}{a}$. The right side is $1 - \dfrac{1}{a^1} = \dfrac{a-1}{a}$. They are equal so the base case is verified.

INDUCTION CASE. Assume that the formula (1) holds for some $n \in \mathbb{N}$. For $n+1$

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{a-1}{a^k} &= \frac{a-1}{a^{n+1}} + \sum_{k=1}^{n} \frac{a-1}{a^k} \\ &= \frac{a-1}{a^{n+1}} + 1 - \frac{1}{a^n} && \text{Using the induction hypothesis (1)} \\ &= 1 + \frac{a-1}{a^{n+1}} - \frac{a}{a^{n+1}} \\ &= 1 - \frac{1}{a^{n+1}}. \end{aligned}$$

We conclude that the formula holds for $n+1$ as well.

Since we have established both the base case and induction case, by mathematical induction, (1) holds for all $n \in \mathbb{N}$. □

2. *Recall the definition given in class.*

Suppose that we have two nonempty sets A and B and a function $f : A \to B$. A function $g : B \to A$ is called an inverse function *of f iff*

(1.) $f(g(y)) = y$ for all $y \in B$;

(2.) $g(f(x)) = x$ for all $x \in A$;

Let $f : A \to B$ be a function and $E \subset B$ a set. Define $f^{-1}(E)$. Suppose that $f : A \to B$ has an inverse function called $g : B \to A$. Let $E \subset B$. Show that $f^{-1}(E) = g(E)$.

The preimage set is defined to be

$$f^{-1}(E) = \{x \in A : f(x) \in E\}.$$

To show $f^{-1}(E) = g(E)$, we first show $f^{-1}(E) \subset g(E)$ and then we show $f^{-1}(E) \supset g(e)$.

To show $f^{-1}(E) \subset g(E)$, we choose an $x \in f^{-1}(E)$ to show that it is in $g(E)$. But by definition of preimage, this means that $f(x) \in E$. Call it $y = f(x) \in E$. Applying g, we have $g(y) \in g(E)$ by the definition of image of g. But by property (2.) of inverse functions, $x = g(f(x)) = g(y)$. Hence $x \in g(E)$ as to be shown.

To show $f^{-1}(E) \supset g(E)$, we choose an $x \in g(E)$ to show that it is in $f^{-1}(E)$. But by definition of image set, this means that there is a $y \in E$ so that $x = g(y)$. Applying f, this means by property (1.) that $y = f(g(y)) = f(x)$. But since $f(x) = y \in E$, this implies by the definition of preimage, that $x \in f^{-1}(E)$, as to be shown. □

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*
STATEMENT 1. *If $f : A \to B$ and $E \subset F \subset A$ are subsets then $f(E) \subset f(F)$.*

TRUE: Choose $y \in f(E)$ to show $y \in f(F)$. Hence there is $x \in E$ so that $f(x) = y$. But $E \subset F$ implies $x \in F$. Thus $y = f(x) \in f(F)$, as to be shown.

STATEMENT 2. *For $E, F \subset X$ any two subsets, $E \backslash F = \emptyset$ implies $E = F$.*

FALSE. Take real subsets $E = [0,1]$ and $F = [0,2]$. Then $E \backslash F = \emptyset$ but $E \neq F$.

STATEMENT 3. *Suppose $f : A \to B$ is a function. Suppose for every $x_1, x_2 \in A$, $f(x_1) \neq f(x_2)$ implies $x_1 \neq x_2$. Then f is one-to-one.*

FALSE. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. The condition is equivalent to its contrapositive "if $x_1 = x_2$ then $f(x_1) = f(x_2)$" which holds for any function, e.g., for f, but f is not one-to-one because $f(-2) = 4 = f(4)$.

4. *Let A, B, C be nonempty sets and $g : A \to B$ and $f : B \to C$ be functions. Write the definition: $f : B \to C$ is onto. Show that if the composite function $f \circ g : A \to C$ is onto then f is onto. Give an example that shows that even if $f \circ g$ is onto, then g does not need to be onto.*

$f : B \to C$ is onto means that for every $z \in C$ there is a $y \in B$ so that $f(y) = z$.

To show that $f : B \to C$ is onto, we choose $z \in C$. Since we assume that $f \circ g : A \to C$ is onto, there is an $x \in A$ so that $f \circ g(x) = z$. Let $y = g(x) \in B$. Then $f(y) = f(g(x)) = f \circ g(x) = z$. Hence we have found a $y \in B$ so that $f(y) = z$. Thus we have shown that f is onto.

Take $A = \{0\}$, $B = \{1, 2\}$ and $C = \{3\}$. Define $g : A \to B$ by $g(0) = 1$. Define $f : B \to C$ by $f(1) = f(2) = 3$. Then g is not onto because $g(A) = \{1\} \neq B$. However $f \circ g$ is onto because $f \circ g(A) = f \circ g(\{0\}) = \{f \circ g(0)\} = \{f(g(0))\} = \{f(1)\} = \{3\} = C$.

5. *Let $E \subset \mathbb{R}$ be a set of real numbers. Suppose that the set is given by*

$$E = \left\{ x \in \mathbb{R} : (\forall \sigma < 1)\ (\exists \tau > 0) \quad \sigma \le x < \sigma + \tau \ \right\}.$$

Write the set E in terms of unions and intersections. Find the complement E^c by negating the expression for E and writing it so that the negators come after the quantifiers. Express E^c in terms of intervals and prove your result.

In terms of intersections and unions,

$$E = \bigcap_{\sigma < 1} \bigcup_{\tau > 0} \Big[\sigma, \sigma + \tau\Big) \qquad \left(\text{which equals } \bigcap_{\sigma < 1} \Big[\sigma, \infty\Big) = \Big[1, \infty\Big). \right)$$

By negating the quantifiers we see that the complement is

$$\begin{aligned} E^c &= \left\{ x \in \mathbb{R} : \quad \sim (\forall \sigma < 1)\ (\exists \tau > 0) \quad \sigma \le x < \sigma + \tau \ \right\} \\ &= \left\{ x \in \mathbb{R} : \quad (\exists \sigma < 1)\ (\forall \tau > 0) \quad (x < \sigma \ \text{ or } \ \sigma + \tau \le x.) \ \right\}. \end{aligned} \tag{2}$$

We expect that $E^c = (-\infty, 1)$. We can check this in several ways, but let us argue with E^c given by formula (2). We first show "$\subset$" and then show "$\supset$."

To show that $E^c \subset (-\infty, 1)$ we choose $x \in E^c$. Then there is $\sigma_0 < 1$ such that $(\forall \tau > 0)(x < \sigma_0 \text{ or } \sigma_0 + \tau \le x)$. It follows that $x < \sigma_0$. If this were not the case and $x \ge \sigma_0$, by taking $\tau_0 > 0$ so large that $\tau_0 > x - \sigma_0$, neither $x < \sigma_0$ nor $\sigma_0 + \tau_0 \le x$ is true so that $(\forall \tau > 0)(x < \sigma_0 \text{ or } \sigma_0 + \tau \le x)$ is false. Thus $x < \sigma_0 < 1$ so $x \in (-\infty, 1)$ as to be shown.

To show that $E^c \supset (-\infty, 1)$ we choose $x \in (-\infty, 1)$ or $x < 1$, and show that $x \in E^c$. If we pick $\sigma_0 = (1 + x)/2$ between x and 1, then $x < \sigma_0$ is true and so $(\forall \tau > 0)(x < \sigma_0 \text{ or } \sigma_0 + \tau \le x)$ is also true for this σ_0. Thus $x \in E^c$.