Math 3210 § 2. **Third Midterm Exam** Name: *Solutions*
Treibergs April 2, 2014

(1.) *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Define: f is uniformly continuous on $\mathbf{R}$. Determine whether $f(x) = \sqrt{1 + |x|}$ is uniformly continuous on $\mathbf{R}$. Prove your answer using just the definition.*

A function $f : \mathbf{R} \to \mathbf{R}$ is *uniformly continuous* iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in \mathbf{R}$ such that $|x - y| < \delta$.

We prove that $f(x) = \sqrt{1 + |x|}$ is uniformly continuous on $\mathbf{R}$. Choose $\epsilon > 0$. Let $\delta = 2\epsilon$. Then for any $x, y \in \mathbf{R}$ such that $|x - y| < \delta$ we have

$$\begin{aligned}
|f(x) - f(y)| &= |\sqrt{1+|x|} - \sqrt{1+|y|}| \\
&= \left| \frac{(\sqrt{1+|x|} - \sqrt{1+|y|})(\sqrt{1+|x|} + \sqrt{1+|y|})}{\sqrt{1+|x|} + \sqrt{1+|y|}} \right| \\
&= \frac{\big|1 + |x| - 1 - |y|\big|}{\sqrt{1+|x|} + \sqrt{1+|y|}} \\
&= \frac{\big||x| - |y|\big|}{\sqrt{1+|x|} + \sqrt{1+|y|}} \\
&\le \frac{|x-y|}{2} < \frac{\delta}{2} = \frac{2\epsilon}{2} = \epsilon
\end{aligned}$$

where we have used the reverse triangle inequality and $\sqrt{1 + |x|} \ge 1$.

(2.) *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

A. Statement: *Let $f_n : \mathbf{R} \to \mathbf{R}$ be a sequence of continuous functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in \mathbf{R}$. Then f is continuous.*

FALSE. Let $f_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ x^n, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } 1 < x. \end{cases}$ Let $f(x) = \begin{cases} 0, & \text{if } x < 1; \\ 1, & \text{if } 1 \le x. \end{cases}$. Then f_n is continuous on $\mathbf{R}$, $f_n \to f$ as $n \to \infty$ pointwise, but f is not continuous.

B. Statement: *Let $f : [0, 1] \to \mathbf{R}$ is a continuous function such that $f(0) = f(1)$. Then there is a $c \in (0, 1)$ such that $f'(c) = 0$.*

FALSE. Consider $f(x) = |x - \frac{1}{2}|$. Then $f'(x) = -1$ if $x < \frac{1}{2}$, $f'(x) = 1$ if $x > \frac{1}{2}$ and the derivative is undefined at $x = \frac{1}{2}$. Hence at no x is the derivative zero.

C. Statement: *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous bounded function. Then for every $y \in \mathbf{R}$ such that $I = \inf_{\mathbf{R}} f < y < \sup_{\mathbf{R}} f = S$ there is an $x \in \mathbf{R}$ so that $f(x) = y$.*

TRUE. Since f is bounded, both I and S are finite. Given $I < y < S$, by the definition if inf and sup there are numbers $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) < y < f(x_2)$. Since f is continuous on the closed interval between x_1 and x_2 and y is an intermediate value, by the Intermediate Value Theorem, there is c between x_1 and x_2 such that $f(c) = y$.

(3.) *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a function and* $a \in \mathbf{R}$*. State the definition:* f *is* differentiable *at* a*. Assume that* $f(x)$ *is differentiable at* a*. Using only the definition, show that the function* $g(x) = xf(x)$ *is also differentiable at* a*.*

f is differentiable at a iff the limit $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists and is finite. Its value is the derivative $f'(a)$.

We compute the limit of difference quotients and use the product rule trick. The limit of difference quotients

$$\begin{aligned}
\lim_{x\to a} \frac{g(x) - g(a)}{x - a} &= \lim_{x\to a} \frac{xf(x) - af(a)}{x - a} \\
&= \lim_{x\to a} \frac{xf(x) - xf(a) + xf(a) - af(a)}{x - a} \\
&= \lim_{x\to a} \left(x\frac{f(x) - f(a)}{x - a} + f(a)\frac{x - a}{x - a} \right) \\
&= \lim_{x\to a} \left(x\frac{f(x) - f(a)}{x - a} + f(a) \right) \\
&= af'(a) + f(a),
\end{aligned}$$

where we have used the limit of a product is the product of limits. Thus the limit of difference quotients of g exists at a and equals $g'(a) = af'(a) + f(a)$, as expected from the product rule!

(4.) *Suppose* $f : \mathbf{R} \to \mathbf{R}$ *is a continuous function and* $a \in \mathbf{R}$*. Assume that* $f(a) \neq 0$*. Then there is a positive* $\eta > 0$ *such that* $|f(x)| \geq \eta$ *whenever* $x \in \mathbf{R}$ *and* $|x - a| < \eta$*.*

Since $f(x)$ is continuous at a, for every $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(a)| < \varepsilon$ whenever $x \in \mathbf{R}$ and $|x - a| < \delta$. Take $\varepsilon_0 = \frac{1}{2}|f(a)| > 0$. There is $\delta_0 > 0$ so that if $|x - a| < \delta_0$ then

$$|f(x) - f(a)| < \varepsilon_0 = \frac{1}{2}|f(a)|.$$

It follows that for the same x,

$$|f(x)| = |f(a) + f(x) - f(a)| \geq |f(a)| - |f(x) - f(a)| > |f(a)| - \frac{1}{2}|f(a)| = \frac{1}{2}|f(a)|.$$

Thus if we put $\eta = \min\{\delta, \frac{1}{2}|f(a)|\} > 0$ then the conclusion follows: $|x - a| < \eta$ implies $|x - a| < \delta$ which implies $|f(x)| > \frac{1}{2}|f(a)| \geq \eta$, as to be shown.

(5.) *Let* $\mathcal{D} \subset \mathbf{R}$ *be a nonempty subset and* $f, f_n : \mathcal{D} \to \mathbf{R}$ *be functions. State the definition:* $f_n \to f$ *as* $n \to \infty$ *uniformly on* $\mathcal{D}$*. Let* $f_n(x) = \dfrac{n}{x + n}$*. Determine whether the sequence* $\{f_n(x)\}$ *converges uniformly on* $\mathcal{D} = (0, 1)$*. Prove your answer using just the definition of uniform convergence.*

We say that $f_n \to f$ as $n \to \infty$ *converges uniformly* on $\mathcal{D}$ iff for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ so that $|f_n(x) - f(x)| < \epsilon$ whenever $x \in \mathcal{D}$ and $n > N$.

To show uniform convergence using the definition, we need to identify the limiting function. However, the limiting function will be the same as the one we get for the pointwise limit which we can find for $x \in (0, 1)$ by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{x + n} = \lim_{n\to\infty} \frac{1}{\frac{x}{n} + 1} = \frac{1}{0 + 1} = 1.$$

Now we show that the convergence $f_n \to f$ as $n \to \infty$ is uniform. Choose $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Then for any number $x \in (0, 1)$ and $n > N$, because $1 > x > 0$ and $x + n > 0$ we have

$$|f_n(x) - f(x)| = \left| \frac{n}{x + n} - 1 \right| = \left| \frac{n - x - n}{x + n} \right| = \frac{x}{x + n} < \frac{1}{0 + n} = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon.$$