Math 3210 § 1. **Final Exam** Name:________________
Treibergs April 27, 2017

This is a closed book test except that you are allowed four *cheat sheets:* a new 8.5" × 11" page of notes plus the ones from the midterms. No other books, papers, calculators, tablets, laptops, phones or other messaging devices are permitted. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.	/21
2.	/21
3.	/21
4.	/21
5.	/21
6.	/22
7.	/22
8.	/22
9.	/22
Total	/150

1. (a) [3] Define: the infinite sum $\sum_{k=1}^{\infty} a_k$ *converges* to a real number S.

(b) [18] Determine if the sum converges to S, be it finite or infinite, and prove your result.

$$S = \sum_{n=1}^{\infty} \frac{1}{n + \frac{1}{2}}$$

Your grades will be posted at my office according to | *Secret Id.* :

2. (a) [3] Let $E \subset \mathbf{R}$ be a nonempty subset. Define the *infimum of* E, $I = \inf E$.

(b) [18] Find the infimum $\inf E$ and prove your result.

$$E = \left\{ \int_0^1 f(x)^2 \, dx \;\middle|\; f : [0,1] \to \mathbf{R} \text{ is continuous and } f(0) = f(1) = 1. \right\}$$

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] If x and y are real numbers then $xy = 0$ implies $x = 0$ or $y = 0$.

TRUE: ○	FALSE: ○

   (b) [7] For integrable $f : [a, b] \to \mathbf{R}$, there is a point $c \in [a, b]$ such that $\displaystyle\int_a^c f(x)\,dx = \int_c^b f(x)\,dx$.

TRUE: ○	FALSE: ○

   (c) [7] If $\{a_i\}$ is a bounded sequence, then it has a Cauchy Subequence.

TRUE: ○	FALSE: ○

4. (a) [3] Let $f : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$. Define: *f is differentiable at c.*

   (b) [18] Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $x = 0$ and that $f(0) \neq 0$. Using just your definition of differentiability and not a differentiability theorem, show that $g(x)$ is differentiable at $x = 0$, where

$$g(x) = \frac{1}{f(x)^2}$$

5. For $n \in \mathbf{N}$, let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions.

   (a) [3] Define: $f_n \to f$ *converges uniformly on* $\mathbf{R}$ *as* $n \to \infty$.

   (b) [18] Let $|b| < 1$. Prove that f_n converges uniformly on $\mathbf{R}$, where

$$f_n(x) = \sum_{k=1}^{n} b^k \cos(k\,x)$$

6. (a) [3] Let $f : [a, b] \to \mathbf{R}$. Define: f is *continuous* at $c \in [a, b]$.

(b) [19] Suppose that $f : [-1, 1] \to \mathbf{R}$ is a continuous function such that $f(x) \ge 0$ for all x and $f(0) > 0$. Show that

$$\int_{-1}^{1} f(x)\,dx > 0.$$

7. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] Let $f, f_n, g : [0,1] \to \mathbf{R}$ be continuous functions. If $f_n \to f$ uniformly on $[0,1]$, then

   $$\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\, dx = \int_0^1 f(x) g(x)\, dx$$

   TRUE: ◯ FALSE: ◯

   (b) [7] If $f : \mathbf{R} \to \mathbf{R}$ is one-to-one and onto then it is continuous.

   TRUE: ◯ FALSE: ◯

   (c) [8] Let $f : (0,1) \to \mathbf{R}$ be a differentiable function with $f'(x) > 0$ for all $x \in (0,1)$. Then f is strictly increasing on $(0,1)$.

   TRUE: ◯ FALSE: ◯

8. Let f be a bounded function on the closed bounded interval $[a, b]$.

   (a) [3] Define what it means for f to be *integrable* on $[a, b]$ and what the *Riemann integral* of f on $[a, b]$ is.

   (b) [3] Complete the statement of the theorem. [Of several possible answers, select the one you prefer for part (c).]

   Theorem. *The Riemann integral of f on $[a, b]$ exists if and only if*

   (c) [16] Using the theorem in (b), show that if for some $L < \infty$ the function $f : [0, 1] \to \mathbf{R}$ satisfies

$$|f(x) - f(y)| \le L\Big|x - y\Big|^{\frac{1}{2}}, \qquad \text{for all } x, y \in [0, 2].$$

   then f is integrable on $[0, 2]$.

9. [22] Prove that the improper integral converges

$$I = \int_{-\infty}^{\infty} \frac{e^{-|t|}}{\sqrt{|t|}}\, dt$$