Math 3210 § 2. **Final Exam** Name: Solutions
Treibergs May 2, 2019

1. *Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a fixed point: there is $c \in [0,1]$ such that $f(c) = c$. Assuming in addition that f is differentiable at x and $|f'(x)| < 1$ for all $x \in (0,1)$, show that the fixed point is unique.*

   The existence of a fixed point is one of the standard applications of the Intermediate Value Theorem. Let $g(x) = f(x) - x$. $g(x)$ is a continuous function on $[0,1]$ because both $f(x)$ and x are continuous. If $g(0) = 0$ then $c = 0$ is the fixed point because $f(0) - 0 = 0$. If $g(1) = 0$ then $c = 1$ is a fixed point since $f(1) - 1 = 0$. Otherwise $f(0) > 0$ and $f(1) < 1$ so $g(0) > 0$ and $g(1) < 0$. Thus $y = 0$ is intermediate between $g(0)$ and $g(1)$. By the Intermediate Value Theorem, there is $c \in (0,1)$ such that $g(c) = 0$. For this c we have $f(c) - c = 0$ so c is the desired fixed point.

   The uniqueness of the fixed point follows from an application of the Mean Value Theorem. Suppose for contradiction that there are two different fixed point $c, d \in [0,1]$ such that $f(c) = c$ and $f(d) = d$. We show that this is impossible under the additional hypotheses. We may suppose $c < d$ by swapping names, if necessary. Then, by assumption, f is continuous on $[c,d]$ because it is a subset of $[0,1]$ and it is differentiable on (c,d) because this is a subset of $(0,1)$. Hence the hypotheses for the Mean Value Theorem hold. It says that there is a $\xi \in (c,d)$ such that

   $$d - c = f(d) - f(c) = f'(\xi)(d - c).$$

   Taking absolute values and estimating,

   $$|d - c| = |f'(\xi)|\,|d - c| < 1 \cdot |d - c|.$$

   As $|d - c| > 0$, this is a contradiction.

2. *Let $f : \mathbf{R} \to \mathbf{R}$. Define the* supremum of f, $\quad S = \sup_{x\in\mathbf{R}} f(x)$. *Find* $S = \sup_{x\in\mathbf{R}} \dfrac{x^2}{x^2+1}$ *and prove your result.*

   The supremum of f is an extended real number S. If f is not bounded above on $\mathbf{R}$, then the supremum $\sup_{x\in\mathbf{R}} f(x) = \infty$. If f is bounded above on $\mathbf{R}$, then $S \in \mathbf{R}$ satisfies two properties:

   (1) S is an upper bound for f: $(\forall x \in \mathbf{R})(f(x) \le S)$, and,

   (2) S is the smallest of upper bounds, or to put it another way, no smaller number is an upper bound: $(\forall \epsilon > 0)(\exists x \in \mathbf{R})(f(x) > S - \epsilon)$.

   We claim that $\sup_{x\in\mathbf{R}} \dfrac{x^2}{x^2+1} = 1$. To see that 1 is an upper bound, we have $x^2 < x^2 + 1$ for all $x \in \mathbf{R}$ so that $\dfrac{x^2}{x^2+1} \le 1$ for all $x \in \mathbf{R}$. To see that there are no smaller upper bounds, choose $\epsilon > 0$. Let $x = \frac{1}{\sqrt{\epsilon}}$. For this x,

   $$f(x) = \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1} = 1 - \frac{1}{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 + 1} = 1 - \frac{1}{\frac{1}{\epsilon} + 1} = 1 - \frac{\epsilon}{1+\epsilon} > 1 - \epsilon.$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) *If $x > 0$ then $x^a x^b = x^{a+b}$ for any $a, b \in \mathbf{R}$.*

True. This fact depends on how functions x^a are defined for arbitrary real numbers, not just integers of rational numbers. We have

$$x^a = \exp(a \log x)$$

so that the desired property follows from the corresponding property of the exponential function which in turn depends on an addition formula for the integral that defines natural logarithm. Indeed

$$x^a x^b = \exp(a \log x) \exp(b \log x) = \exp(a \log x + b \log x) = \exp([a + b] \log x) = x^{a+b}.$$

(b) *For $f : [-1, 0) \cup (0, 1] \to \mathbf{R}$, such that f is integrable on $[-1, -\epsilon]$ and on $[\epsilon, 1]$ for every $0 < \epsilon \le 1$, suppose $\lim_{\epsilon \to 0+} \left(\int_{-1}^{-\epsilon} f(t)\,dt + \int_{\epsilon}^{1} f(t)\,dt \right) = 0$. Then the improper integral $\int_{-1}^{1} f(t)\,dt$ exists and equals zero.*

False. The limit is a Cauchy Principal Value which may exist without the function being improperly integrable. For example if $f(x) = x^{-3}$ then for $0 < \epsilon \le 1$ we have

$$\int_{-1}^{-\epsilon} \frac{dt}{t^3} + \int_{\epsilon}^{1} \frac{dt}{t^3} = \left[-\frac{1}{2t^2} \right]_{-1}^{-\epsilon} + \left[-\frac{1}{2t^2} \right]_{\epsilon}^{1} = \left[-\frac{1}{2\epsilon^2} + \frac{1}{2} \right] + \left[-\frac{1}{2} + \frac{1}{2\epsilon^2} \right] = 0$$

so the limit is zero but the function $f(t) = t^{-3}$ is not improperly integrable. For the improper integral $\int_{-1}^{1} \frac{dt}{t^3}$ to exist, both limits to the left and right of zero have to exist by themselves, but neither do.

$$\lim_{\epsilon \to 0+} \int_{-1}^{-\epsilon} \frac{dt}{t^3} = \lim_{\epsilon \to 0+} \left[-\frac{1}{2t^2} \right]_{-1}^{-\epsilon} = \lim_{\epsilon \to 0+} \left[-\frac{1}{2\epsilon^2} + \frac{1}{2} \right] = -\infty,$$

$$\lim_{\delta \to 0+} \int_{\delta}^{1} \frac{dt}{t^3} = \lim_{\delta \to 0+} \left[-\frac{1}{2t^2} \right]_{\delta}^{1} = \lim_{\delta \to 0+} \left[-\frac{1}{2} + \frac{1}{2\delta^2} \right] = \infty.$$

(c) *If $f : \mathbf{R} \to \mathbf{R}$ is differentiable at all points, then $f'(x)$ is continuous for all $x \in \mathbf{R}$.*

False. The differentiability of $f(x)$ at $x = a$ implies the continuity of f at $x = a$, but it says nothing about the continuity of the derivative. Here is the example discussed in class:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \ne 0; \\ 0, & \text{if } x = 0. \end{cases}$$

For $x \ne 0$ this is the composition of differentiable functions, so differentiable. At $x = 0$ we use the fact that $|f(x)| \le x^2$ for all x so f is stuck between a rock and a hard place. The difference quotients at zero satisfy

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \frac{|f(x)|}{|x|} \le \frac{x^2}{|x|} = |x|$$

which tends to zero as $x \to 0$ so that f is differentiable at zero and $f'(0) = 0$. Thus f is differentiable at all points. However the derivative is not continuous at zero. Computing the derivative at $x \ne 0$ we see that

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

which does not have a limit as $x \to 0$ so it doesn't converge to $f'(0)$. Thus f' is not continuous at $x = 0$.

4. *Define: the real sequence* $\{a_n\}$ *is a* Cauchy Sequence. *Show that the sequence* $\{a_n\}$ *converges to a real number,* $a_n \to L$ *as* $n \to \infty$ *where* a_n *is defined recursively by starting with* $a_1, a_2 \in \mathbf{R}$ *any two real numbers and*

$$a_n = \frac{a_{n-1} + a_{n-2}}{2}, \qquad \text{for all } n \geq 3.$$

The sequence is a *Cauchy Sequence* if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ so that

$$|a_m - a_n| < \epsilon \qquad \text{whenever } m, n \in \mathbb{N} \text{ are such that } m > N \text{ and } m > N.$$

We show that the given $\{a_n\}$ is a Cauchy Sequence, thus convergent. To do this, we establish the recursion for the difference of consecutive terms, as in the homework problem. Thus for any $n \geq 2$ we have

$$a_{n+1} - a_n = \frac{a_n + a_{n-1}}{2} - a_n = -\frac{1}{2}(a_n - a_{n-1}).$$

It follows by induction that for every $n \geq 1$ we have

$$a_{n+1} - a_n = \left(-\frac{1}{2}\right)^{n-1} (a_2 - a_1)$$

so that for every $n \geq 1$ we have

$$|a_{n+1} - a_n| \leq \left(\frac{1}{2}\right)^{n-1} |a_2 - a_1|.$$

Now it follows that $\{a_n\}$ is a Cauchy Sequence. Choose $\epsilon > 0$. Let N be so large that $(\frac{1}{2})^{N-2}|a_2 - a_1| < \epsilon$. Then for any $m, n > N$ we have either $m = n$ so $|a_m - a_n| = 0 < \epsilon$ or $n \neq m$. By swapping names if necessary, we may assume that $m > n$. In this case, by constructing the telescoping sum,

$$\begin{aligned}
|a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq \left\{ \left(\frac{1}{2}\right)^{m-2} + \left(\frac{1}{2}\right)^{m-3} + \cdots + \left(\frac{1}{2}\right)^{n-1} \right\} |a_2 - a_1| \\
&= \left(\frac{1}{2}\right)^{n-1} \sum_{k=0}^{m-n-1} \left(\frac{1}{2}\right)^k |a_2 - a_1| \\
&= \left(\frac{1}{2}\right)^{n-1} \frac{1 - (\frac{1}{2})^{m-n}}{1 - \frac{1}{2}} |a_2 - a_1| \\
&\leq \left(\frac{1}{2}\right)^{n-2} |a_2 - a_1| \\
&< \left(\frac{1}{2}\right)^{N-2} |a_2 - a_1| < \epsilon.
\end{aligned}$$

As a curiosity, which is not part of the answer, we can compute the limit.

$$
\begin{aligned}
L &= \lim_{n\to\infty} a_n \\
&= \lim_{n\to\infty} \{(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) + a_1\} \\
&= \lim_{n\to\infty} \left\{ a_1 + \sum_{k=2}^{n} (a_k - a_{k-1}) \right\} \\
&= \lim_{n\to\infty} \left\{ a_1 + \sum_{k=2}^{n} \left(-\frac{1}{2}\right)^{k-2} (a_2 - a_1) \right\} \\
&= \lim_{n\to\infty} \left\{ a_1 + \sum_{j=0}^{n-2} \left(-\frac{1}{2}\right)^{j} (a_2 - a_1) \right\} \\
&= \lim_{n\to\infty} \left\{ a_1 + \frac{1 - (-\frac{1}{2})^{n-1}}{1 - (-\frac{1}{2})} (a_2 - a_1) \right\} \\
&= a_1 + \frac{2}{3}(a_2 - a_1) = \frac{1}{3}a_1 + \frac{2}{3}a_2.
\end{aligned}
$$

Thus L is an average of the starting numbers, as we might have expected.

5. *Let $f, f_n : \mathbf{R} \to \mathbf{R}$ be functions for $n \in \mathbf{N}$. Define: $f_n \to f$* converges uniformly *on $\mathbf{R}$ as $n \to \infty$. Prove that the sequence $\{f_n\}$ converges uniformly on $\mathbf{R}$ as $n \to \infty$, where*

$$f_n(x) = \frac{x}{1 + nx^2}.$$

$f_n \to f$ *converges uniformly* on $\mathbf{R}$ if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ so that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } n > N.$$

Sketch the functions! Here is the graph using Macintosh's Grapher.

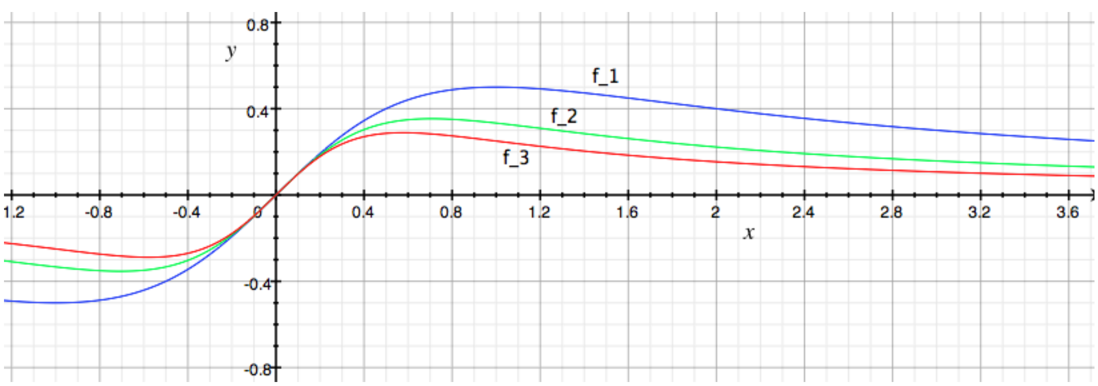

Figure 1: Sketch of f_1, f_2 and f_3.

We have $f_n(0) = 0$ for all n and if $x \neq 0$ then $\lim\limits_{n\to\infty} f_n(x) = 0$ so f_n converges to $f = 0$ pointwise on $\mathbf{R}$. If it converges uniformly then the limiting function has to be the same $f = 0$.

We claim that $|f_n(x)-0| \le \dfrac{1}{\sqrt{n}}$ for all x and n so that $f_n \to 0$ uniformly by the Weierstrass M-test. We give two proofs of the claim.

First proof of the claim uses calculus. The function $f_n(x)$ is odd so we only need to prove it for $x \ge 0$. We note that $\lim\limits_{x\to\infty} f_n(x) = 0$ and $f_n(x) > 0$ for $0 < x < \infty$ so there is an interior maximum. Differentiating,

$$\frac{d}{dx} f_n(x) = \frac{(1+nx^2) - x\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

It is zero only when $x = \pm\dfrac{1}{\sqrt{n}}$ so there is only one max. Thus

$$|f_n(x)-0| \le \left| f_n\left(\pm\frac{1}{\sqrt{n}}\right)\right| = \frac{\left|\pm\frac{1}{\sqrt{n}}\right|}{1+n\left(\pm\frac{1}{\sqrt{n}}\right)^2} = \frac{1}{2\sqrt{n}} \le \frac{1}{\sqrt{n}}.$$

The second proof does the estimate one way for small x and another for large x. Indeed, if $|x| \le \dfrac{1}{\sqrt{n}}$ then

$$|f_n(x)-0| = \frac{|x|}{1+nx^2} \le \frac{\frac{1}{\sqrt{n}}}{1+0} = \frac{1}{\sqrt{n}}.$$

If $|x| > \dfrac{1}{\sqrt{n}}$ then $\dfrac{1}{|x|} < \sqrt{n}$ so that

$$|f_n(x)-0| = \frac{|x|}{1+nx^2} = \frac{\frac{1}{|x|}}{\frac{1}{x^2}+n} < \frac{\sqrt{n}}{0+n} = \frac{1}{\sqrt{n}}.$$

In both cases,

$$|f_n(x)-0| \le \frac{1}{\sqrt{n}}$$

as claimed.

6. *Let* $f : \mathbf{R} \to \mathbf{R}$. *Define:* f *is* uniformly continuous *on* $\mathbf{R}$. *Suppose that* $|f(u_n)-f(v_n)| \to 0$ *as* $n \to \infty$ *for any pair of real sequences such that* $|u_n - v_n| \to 0$ *as* $n \to \infty$. *Show that* f *is uniformly continuous on* $\mathbf{R}$.

   f is *uniformly continuous* on $\mathbf{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that

   $$|f(u) - f(v)| < \epsilon \qquad \text{whenever } u, v \in \mathbf{R} \text{ are such that } |u-v| < \delta.$$

   The condition gives a sequential characterization of uniform continuity. Its proof is almost the same as the proof that the sequential condition for continuity at a point a implies continuity at a.

   One proves the contrapositive statement: if f is not uniformly continuous on $\mathbf{R}$ then the sequential condition does not hold. The negation of uniform continuity is: there is $\epsilon_0 > 0$ such that for every $\delta > 0$ there are $u_\delta, v_\delta \in \mathbf{R}$ such that $|u_\delta - v_\delta| < \delta$ but $|f(u_\delta)-f(v_\delta)| \ge \epsilon_0$. Take $\delta = \frac{1}{n}$. Then there are sequences $u_n, v_n \in \mathbf{R}$ such that

   $$|u_n - v_n| < \frac{1}{n}$$

so that $|u_n - v_n| \to 0$ as $n \to \infty$ but

$$|f(u_n) - f(v_n)| \geq \epsilon_0$$

so that $|f(u_n) - f(v_n)|$ does not converge to zero as $n \to \infty$. In other words, it is not the case that $|f(u_n) - f(v_n)| \to 0$ as $n \to \infty$ for any pair of real sequences such that $|u_n - v_n| \to 0$ as $n \to \infty$.

7. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT. *Let f, g be differentiable on $(-1, 1)$ such that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (0, 1)$. If $\lim_{x \to 0+} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to 0+} \frac{f(x)}{g(x)} = L$.*

   FALSE. All of the hypotheses of l'Hôpital's Rule are not met. *e.g.,* taking $f(x) = 2 + x$, $g(x) = 3 + x$ we have f and g differentiable, g and g' nonzero on $(0, 1)$ with

$$\lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{1}{1} = 1 \quad \text{but} \quad \lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{2 + x}{3 + x} = \frac{2}{3}.$$

   (b) STATEMENT. *If $f : \mathbf{R} \to \mathbf{R}$ is differentiable at 0 and $f'(0) > 0$ then there is a $\delta > 0$ such that $f(x) > f(0)$ whenever $0 < x < \delta$.*

   TRUE. Use the definition of differentiable: the limit exists and equals $f'(0) \in \mathbf{R}$:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0).$$

   Apply the $\epsilon - \delta$ definition of limit: for every $\epsilon > 0$ there is a $\delta > 0$ so that

$$\left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right| < \epsilon \qquad \text{whenever } x \in \mathbf{R} \text{ such that } 0 < |x - 0| < \delta.$$

   In particular, if we choose $\epsilon = f'(0)$ and take the corresponding $\delta > 0$ we have

$$\frac{f(x) - f(0)}{x - 0} - f'(0) > -\epsilon \qquad \text{if } 0 < |x - 0| < \delta.$$

   In particular

$$\frac{f(x) - f(0)}{x - 0} > f'(0) - \epsilon = 0 \qquad \text{if } 0 < x < \delta.$$

   Thus $f(x) - f(0) > 0$ if $0 < x < \delta$.

   (c) STATEMENT. *If $f : [0, 1] \to \mathbf{R}$ is integrable, then $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$ for all $x \in (0, 1)$.*

   FALSE. In the Fundamental Theorem of Calculus II, the integral is differentiable only at points of continuity of f. So to answer the question, we need to construct a counterexample. Let

$$f(x) = \begin{cases} -1, & \text{if } x \leq \frac{1}{2}; \\ 1, & \text{if } x > \frac{1}{2}. \end{cases}.$$

   Then f is integrable and

$$F(x) = \int_0^x f(t)\, dt = \left| x - \frac{1}{2} \right| - \frac{1}{2}$$

   which is not differentiable at $x = \frac{1}{2}$.

8. *Let f be a bounded function on the closed bounded interval $[a, b]$. Define what it means for f to be integrable on $[a, b]$ and what the Riemann integral of f on $[a, b]$ is. Complete the statement of the theorem.*

[Of several possible answers, select the one you prefer for the third part of the problem.]

A function f is *integrable* if its upper integral equals its lower integral

$$\overline{\int_a^b} f(t)dt = \underline{\int_a^b} f(t)dt.$$

The integral $\int_a^b f(t)dt$ is then defined to be their common value. The upper and lower integrals are defined to be

$$\overline{\int_a^b} f(t)dt = \inf_P U(f, P), \qquad \underline{\int_a^b} f(t)dt = \sup_P L(f, P),$$

where inf and sup are taken over all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ and where the upper and lower sums are

$$U(f, P) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}), \qquad L(f, P) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}),$$

where $I_k = [x_{k-1}, x_k]$ is the kth interval of P and

$$M_k(f) = \sup_{I_k} f, \qquad m_k(f) = \inf_{I_k} f.$$

Theorem. The Riemann integral of f on $[a, b]$ exists if and only if

for every $\epsilon > 0$ there is a partition P of $[a, b]$ such that $$U(f, P) - L(f, P) < \epsilon.$$

Let $0 \le a_n \le 1$ be a sequence such that $a_n \to 0$ as $n \to \infty$. Using the theorem above, show that f is integrable on $[0, 1]$, where

$$f(x) = \begin{cases} 1, & \text{if } x = a_n \text{ for some } n \in \mathbb{N}; \\ 0, & \text{if } x \neq a_n \text{ for all } n \in \mathbb{N}. \end{cases}$$

Draw the picture!

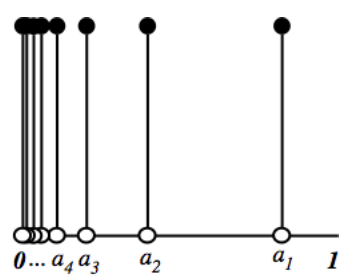

Figure 2: Sketch of function.

The function is discontinuous at every a_i and at 0. The idea is to take a partition that lumps infinitely many a_i's in the subinterval $[0, \delta]$ and then surrounds the finitely many jumps at the remaining a_i's by a tiny intervals $[a_i - \eta, a_i + \eta]$. $M_k(f) - m_k(f) = 1$ for these intervals and is zero for all the others, making the total sum small.

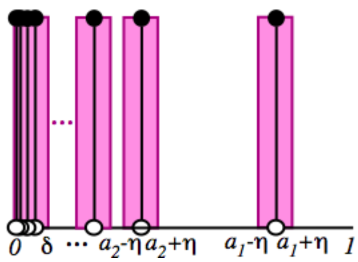

Figure 3: Intervals where $M_k(f) - m_k(f) = 1$.

Choose $\epsilon > 0$. Choose $0 < \delta < \frac{\epsilon}{2}$ such that $\delta \neq a_i$ for all i. Since $a_i \to 0$ there are only finitely many a_i's greater than δ. Call them $a_{i_1}, a_{i_2}, \ldots, a_{i_J}$. Now pick η so small that $\eta < \dfrac{\epsilon}{4J+1}$ and such that all of the intervals

$$\left[a_{i_j} - \eta, a_{i+j} + \eta\right]$$

either coincide (in case some $a_{i_j} = a_{i_{j'}}$) or are pairwise disjoint from each other and disjoint from $[0, \delta]$. Let the partition be

$$P = \{0, \delta, 1\} \cup \left\{a_{i_j} - \eta, a_{i+j} + \eta\right\}_{j=1,\ldots J}$$

It follows for all intervals of the form $I_k = [0, \delta]$ or $I_k = [a_{i_j} - \eta, a_{i_j} + \eta]$ we have $f(a_i) = 1$ and $f(x) = 0$ for points close to a_i so $M_k(f) - m_k(f) = 1$. For all other intervals like $I_k = [\delta, a_{i_J}]$ or $I_k = [a_{i_j} + \eta, a_{i_{j+1}} - \eta]$, the function is dead zero, so that $M_k(f) - m_k(f) = 0$ for this second type or interval. Put $\Delta_k = \text{length}(I_k)$. It follows that

$$\begin{aligned}
U(f,p) - L(f,p) &= \sum_{I_k \text{ is type I}} \left(M_k(f) - m_k(f)\right)\Delta_k + \sum_{I_k \text{ is type II}} \left(M_k(f) - m_k(f)\right)\Delta_k \\
&= \sum_{I_k = [0,\delta]} \left(M_k(f) - m_k(f)\right)\Delta_k + \sum_{I_j = [a_{i_j} - \eta, a_{i_j} + \eta]} \left(M_j(f) - m_j(f)\right)\Delta_j + 0 \\
&= 1 \cdot \delta + J \cdot 1 \cdot 2\eta \\
&< \frac{\epsilon}{2} + \frac{2J\epsilon}{4J+1} < \epsilon.
\end{aligned}$$

By the boxed theorem, f is integrable on $[0, 1]$.

9. *Suppose that* $g : [a, b] \to \mathbf{R}$ *is an integrable function on a closed bounded interval. Show that*

$$\lim_{x \to b-} \int_a^x g(t)\, dt = \int_a^b g(t)\, dt.$$

This problem shows that an integrable function is also improperly integrable.

Since g is integrable on $[a, b]$, it is bounded: there is $M \in \mathbf{R}$ such that $|g(x)| \leq M$ for all $x \in [a, b]$. Integrable also implies for every $a \leq x \leq b$ we have

$$\int_a^x g(t)\, dt + \int_x^b g(t)\, dt = \int_a^b g(t)\, dt.$$

It follows that

$$\begin{aligned} \left| \int_a^b g(t)\, dt - \int_a^x g(t)\, dt \right| &= \left| \int_x^b g(t)\, dt \right| \\ &\leq \int_x^b |g(t)|\, dt \\ &\leq \int_x^b M\, dt = M(b - x) \end{aligned}$$

which tends to zero as $x \to b-$.