Math 3220 § 1. **Second Midterm Exam** Name: Solutions

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1. Let $E \subseteq \mathbf{R}^2$.

   (a) [3] Define: E is an *open* set.

   (b) [19] Using just the definition, show that $E = \{(x, y) \in \mathbf{R}^2 : 1 < x < 2\}$ is an open set.

2. Let $f : \mathbf{R}^2 \to \mathbf{R}$ and $(x_0, y_0) \in \mathbf{R}^2$.

   (a) [3] Define: f is continuous at (x_0, y_0).

   (b) [19] Using just the definition, show that $f(x, y) = xy^2$ is continuous at (x_0, y_0).

3. Let $E \subseteq \mathbf{R}^p$, $f_n : E \to \mathbf{R}^q$ for all $n \in \mathbf{N}$ and $f : E \to \mathbf{R}^q$.

   (a) [3] State the definition: f_n *converges uniformly* to f on E.

   (b) [19] Let
   $$f_n(x, y) = \frac{1}{1 + (x - n)^2 + y^2}.$$
   Then for any $(x, y) \in \mathbf{R}^2$,
   $$\lim_{n\to\infty} f_n(x, y) = 0.$$
   Determine whether $f_n \to 0$ uniformly on $\mathbf{R}^2$ and prove your answer.

UNIFORMLY CONVERGENT: ○	NOT UNIFORMLY CONVERGENT: ○

4. (a) [3] State the definition: $K \subseteq \mathbf{R}^p$ is a *compact set.*

   (b) [19] Let $E \subseteq \mathbf{R}^p$ be an infinite set. Suppose that every point of E is isolated: for every $\mathbf{x} \in E$ there is a $\delta > 0$ so that the only element of E that is in the open δ-ball about $\mathbf{x}$ is $\mathbf{x}$ itself: $(\forall \mathbf{x} \in E)(\exists \delta > 0)(B_\delta(\mathbf{x}) \cap E = \{\mathbf{x}\})$. Show that E is not compact.

5. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [8] Suppose $f : \mathbf{R}^p \to \mathbf{R}$ is continuous. Then $E = \{x \in \mathbf{R}^p : f(x) \le 0\}$ is a closed set.

TRUE: ○	FALSE: ○

   (b) [8] Suppose $f : \mathbf{R}^p \to \mathbf{R}^q$ is continuous. Then $E = \{x \in \mathbf{R}^p : \|f(x)\| \le 1\}$ is a connected set.

TRUE: ○	FALSE: ○

   (c) [8] Let $E \subseteq \mathbf{R}^p$ and $\{\mathbf{x}_k\}$ be a sequence in the boundary ∂E. If the sequence converges to a point $\mathbf{x}_k \to \mathbf{x}$ in $\mathbf{R}^p$, then $\mathbf{x} \in \partial E$.

TRUE: ○	FALSE: ○

Solutions.

1. (a) *Definition:* $E \subseteq \mathbf{R}^p$ is an *open* set if for every $\mathbf{x} \in E$ there is a $\delta > 0$ so that the entire δ-ball about $\mathbf{x}$ is in E: $(\forall \mathbf{x} \in E)(\exists \delta > 0)(B_\delta(\mathbf{x}) \subseteq E)$.

   (b) *Theorem.* $E = \{(x,y) \in \mathbf{R}^2 : 1 < x < 2\}$ is an open set.

   Proof. Choose $(x_0, y_0) \in E$. Let $\delta = \min\{x_0 - 1, 2 - x_0\}$. (This $\delta > 0$ is the distance from (x_0, y_0) to ∂E.) To show $B_\delta((x_0,y_0)) \subseteq E$, choose $(u,v) \in B_\delta((x_0,y_0))$ which means $\|(x_0,y_0) - (u,v)\| < \delta$. We need to conclude $1 < u < 2$ so $(u,v) \in E$. Because $\delta \le x_0 - 1$, we have $u = x_0 - (x_0 - u) \ge x_0 - \|(x_0,y_0) - (u,v)\| > x_0 - \delta \ge x_0 - (x_0 - 1) = 1$. Also because $\delta \le 2 - x_0$ we have $u = x_0 - (x_0 - u) \le x_0 + \|(x_0,y_0) - (u,v)\| < x_0 + \delta \le x_0 + (2 - x_0) = 2$. Thus $1 < u < 2$ so $B_\delta((x_0,y_0)) \subseteq E$.

2. (a) *Definition:* $f : \mathbf{R}^2 \to \mathbf{R}$ is *continuous* at $(x_0,y_0) \in \mathbf{R}^2$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x_0,y_0) - f(u,v)| < \varepsilon$ whenever $(u,v) \in \mathbf{R}^2$ and $\|(x_0,y_0) - (u,v)\| < \delta$.

   (b) *Theorem.* $f(x,y) = xy^2$ is continuous at (x_0,y_0).

   Proof. Choose $\varepsilon > 0$. Let $\delta = \min\{1, \frac{1}{3}\left(1 + 2\|(x_0,y_0)\|\right)^{-2} \varepsilon\}$. Then if $(u,v) \in \mathbf{R}^2$ such that $\|(u,v) - (x_0,y_0)\| < \delta$ we have $|v| \le |v - y_0| + |y_0| \le \|(u,v) - (x_0,y_0)\| + |y_0| < \delta + |y_0| \le 1 + |y_0|$ because $\delta \le 1$. Hence also $|v + y_0| \le |v| + |y_0| \le 1 + 2|y_0|$. Thus, for such (u,v),

$$\begin{aligned}
|f(u,v) - f(x_0,y_0)| &= |uv^2 - x_0 y_0^2| \\
&= |uv^2 - x_0 v^2 + x_0 v^2 - x_0 y_0^2| \\
&\le |u - x_0|\,|v|^2 + |x_0|\,|v + y_0|\,|v - y_0| \\
&\le \left((1 + |y_0|)^2 + |x_0|\,(1 + 2|y_0|)\right)\,\|(u,v) - (x_0,y_0)\| \\
&< 2\left(1 + 2\|(x_0,y_0)\|\right)^2 \delta \\
&\le \varepsilon.
\end{aligned}$$

3. Let $E \subseteq \mathbf{R}^p$, $f_n : E \to \mathbf{R}^q$ for all $n \in \mathbf{N}$ and $f : E \to \mathbf{R}^q$.

   (a) *Definition:* f_n *converges uniformly* to f on E if for every $\varepsilon > 0$ there is a $N \in \mathbf{N}$ so that for all $n > N$ and all $\mathbf{x} \in E$ we have $\|f_n(\mathbf{x}) - f(\mathbf{x})\| < \varepsilon$.

   (b) *Theorem.* $f_n(x,y) = \dfrac{1}{1 + (x-n)^2 + y^2}$ converges pointwise to $f = 0$ on $\mathbf{R}^2$ but not uniformly.

   Proof. Since $f_n = (1 + \|(x,y) - (n,0)\|^2)^{-1}$, we see that if $n > 2\|(x,y)\|$ then $\|(x,y) - (n,0)\| \ge \|(n,0)\| - \|(x,y)\| > \frac{n}{2}$ so that then $|f_n(x,y) - 0| \le (1 + \frac{1}{4}n^2)^{-1} \to 0$ as $n \to \infty$. Thus $f_n \to 0$ pointwise on $\mathbf{R}^2$.

   But the convergence is not uniform. The negation of $f_n \to 0$ uniformly on $\mathbf{R}^2$ is: there is $\varepsilon_0 > 0$ so that for all $N \in \mathbf{N}$ there is $n > N$ and $(u,v) \in \mathbf{R}^2$ so that $|f_n(u,v) - 0| \ge \varepsilon_0$. Let $\varepsilon = \frac{1}{2}$. Choose $N \in \mathbf{N}$. Let $n = N + 1$ and $(u,v) = (n,0)$. Then for this n and (u,v), $|f_n(u,v) - 0| = |1 - 0| = 1 \ge \varepsilon_0$.

4. (a) *Definition:* $K \subseteq \mathbf{R}^p$ is a *compact set* if every open cover of K has a finite subcover. That is, if $\{U_\alpha\}_{\alpha \in A}$ is any collection of open sets of $\mathbf{R}^p$ such that $K \subseteq \cup_{\alpha \in A} U_\alpha$ then there are finitely many subscripts $\{\alpha_1, \ldots, \alpha_n\} \subseteq A$ such that $K \subseteq \cup_{i=1}^n U_{\alpha_i}$.

   (b) *Theorem.* Let $E \subseteq \mathbf{R}^p$ be an infinite set. Suppose that every point of E is isolated: for every $\mathbf{x} \in E$ there is a $\delta > 0$ so that the only element of E that is in the open δ-ball about $\mathbf{x}$ is $\mathbf{x}$ itself: $(\forall \mathbf{x} \in E)(\exists \delta > 0)(B_\delta(\mathbf{x}) \cap E = \{\mathbf{x}\})$. Then E is not compact.

Proof. We exhibit an open cover of E which does not have a finite subcover, thus E fails to be compact. For each point $\mathbf{x} \in E$ let $\delta(\mathbf{x}) > 0$ be the radius of the isolation neighborhood. That is $E \cap B_{\delta(\mathbf{x})}(\mathbf{x}) = \{\mathbf{x}\}$. Consider the collection of open sets $\{B_{\delta(\mathbf{x})}(\mathbf{x})\}_{\mathbf{x}\in E}$. It is a cover of E since if $\mathbf{x} \in E$ then $\mathbf{x} \in B_{\delta(\mathbf{x})}(\mathbf{x})$. Hence $E \subseteq \cup_{\mathbf{x}\in E} B_{\delta(\mathbf{x})}(\mathbf{x})$. But it has no finite subcover. Otherwise there are finitely many $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq E$ such that $E \subseteq \cup_{i=1}^n B_{\delta(\mathbf{x}_i)}(\mathbf{x}_i)$. But since E is infinite, there is $\mathbf{y} \in E - \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ which is not one of the $\mathbf{x}_i$'s. Hence $\mathbf{y} \notin B_{\delta(\mathbf{x}_i)}(\mathbf{x}_i)$ for all $i = 1, \ldots, n$. Thus $\mathbf{y} \notin \cup_{i=1}^n B_{\delta(\mathbf{x}_i)}(\mathbf{x}_i)$, which is a contradiction.

5. (a) *Statement.* Suppose $f : \mathbf{R}^p \to \mathbf{R}$ is continuous. Then $E = \{x \in \mathbf{R}^p : f(x) \le 0\}$ is a closed set.

   TRUE! $C = (-\infty, 0]$ is a closed interval. Thus $E = f^{-1}(C)$ is closed because continuous functions pull back closed sets to closed sets.

   (b) *Statement.* Suppose $f : \mathbf{R}^p \to \mathbf{R}^q$ is continuous. Then $E = \{x \in \mathbf{R}^p : \|f(x)\| \le 1\}$ is a connected set.

   FALSE! Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2 - 2$ which is polynomial, hence continuous. Then $E = f^{-1}([-1, 1]) = [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$ which is not connected.

   (c) *Statement.* Let $E \subseteq \mathbf{R}^p$ and $\{\mathbf{x}_k\}$ is a sequence in the boundary ∂E. If the sequence converges to a point $\mathbf{x}_k \to \mathbf{x}$ in $\mathbf{R}^p$, then $\mathbf{x} \in \partial E$.

   TRUE! The boundary $\partial E = \overline{E} - E^0 = \overline{E} \cap (E^0)^c$ is closed because it is the intersection of the closure, which is a closed set, and the complement of the interior, which is a closed set because it is the complement of an open set. But a closed set contains limits of sequences from the set.