Math 3220 § 1. **Third Midterm Exam** Name: Solutions

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(1a.) *Define: $f : \mathbf{R}^p \to \mathbf{R}^q$ is differentiable at $\mathbf{a} \in \mathbf{R}^p$.*

f is differentiable at $\mathbf{a}$ is there is a $q \times p$ matrix M such that $\lim_{\mathbf{h}\to 0} \frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-M\mathbf{h}}{\|\mathbf{h}\|} = 0$.

(1b.) *Determine whether* $f(x,y) = \begin{cases} \frac{y^5}{x^4+y^4}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$ *is differentiable at* $(0,0)$.

f is not differentiable at $(0,0)$. If the function were differentiable, then the differential $df((0,0)) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial x}(0,0)\right)$ where $\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$ and $\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,0+t)-f(0,0)}{t} = \lim_{t\to 0} \frac{t^5/t^4-0}{t} = 1$. To see that the affine function does not well-approximate, condsider the defining limit

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - df((0,0))\binom{h}{k}}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{\frac{k^5}{h^4+k^4} - 0 - (0,1)\binom{h}{k}}{\sqrt{h^2+k^2}}$$
$$= \lim_{(h,k)\to(0,0)} \frac{k^5 - k(h^4+k^4)}{(h^4+k^4)\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{-h^4 k}{(h^4+k^4)\sqrt{h^2+k^2}}.$$

This limit does not exist so the affine function does not well approximate and the differential does not exist. To see this, consider the approach $(h,k) = (t,0)$ as $t \to 0$. For this path, the limit is 0. On the other hand, for $(h,k) = (t,t)$ the limit is $-1/(2\sqrt{2})$. Since these are inconsistent, the two-dimensional limit does not exist.

(2) *Let $F : \mathbf{R}^4 \to \mathbf{R}^2$ be given by $F(x,y,z,w) = (f_1(x,y,z,w), f_2(x,y,z,w))$ where $f_1(x,y,z,w) = x+yz$ and $f_2(x,y,z,w) = x+y+z+w$. Suppose there is an open set $U \subseteq \mathbf{R}^2$ with $\mathbf{a} = (1,2) \in U$ and a continuously differentiable function $G : U \to \mathbf{R}^2$ with $G(u,v) = \big(g_1(u,v), g_2(u,v)\big)$ and $G(\mathbf{a}) = (3,4)$ such that $f\big(u,v,g_1(u,v),g_2(u,v)\big) = (7,10)$ for all $(u,v) \in U$. Find $dG(\mathbf{a})$.*

All functions are polynomials, thus the Chain Rule applies. Let $\mathcal{G}(u,v) = (u,v,g_1(u,v),g_2(u,v))$ so $\mathcal{G}(1,2) = (1,2,3,4)$. Then differentiate $F \circ \mathcal{G}(u,v) = (7,10)$ using the Chain Rule. One gets $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = dF(\mathcal{G}(1,2)) \circ d\mathcal{G}(1,2)$. Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix}$$

Rewriting and solving, we get at $(u,v) = (1,2)$ and $(x,y,z,w) = (1,2,3,4)$,

$$\begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{pmatrix} = -\begin{pmatrix} \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = -\begin{pmatrix} y & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z \\ 1 & 1 \end{pmatrix}$$
$$= -\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(3.) *Let $f : \mathbf{R}^p \to \mathbf{R}^q$ be such that $f(0) = 0$. Suppose that f and its first partial derivatives exist and are differentiable at all points. Suppose also that the second differential $d^2 f(\mathbf{x})(\mathbf{h})^2 = 0$ for all $\mathbf{h}, \mathbf{x} \in \mathbf{R}^p$. Show that f is a linear function.*

Since the function and its partial derivatives up to first order exist and are differentiable at all points in $\mathbf{R}^p$, we may use the Taylor's approximation up to first order. For all $\mathbf{h} \in \mathbf{R}^p$ there is an interior point in the line segment $\mathbf{c} \in [0, \mathbf{h}]$ such that

$$f(0+\mathbf{h}) = f(0) + df(0)(\mathbf{h}) + \frac{1}{2}d^2f(\mathbf{c})(\mathbf{h})^2 = 0 + df(0)(\mathbf{h}) + 0 = M\mathbf{h},$$

where we have used $f(0) = 0$, $d^2f(\mathbf{c}) = 0$ and let the $q \times p$ matrix $M = df(0)$. Thus, $f(\mathbf{h}) = M\mathbf{h}$ for all $\mathbf{h}$ which is a linear function.

(4.) *Let* $U = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 > 0, x_2 > 0, \ldots, x_p > 0\}$. *Find the maximum of* $\phi(x_1, \ldots, x_p) = \prod_{i=1}^p x_i$ *on the set of points* $(x_1, x_2, \ldots, x_p) \in U$ *that satisfy* $f(x_1, \ldots, x_p) = \sum_{i=1}^p x_i = 1$.

The functions are polynomial so everywhere $\mathcal{C}^1$. Since the logarithm is increasing, we may as well look for the maximum of $\ln \phi(x_1, \ldots, x_p) = \sum_{i=1}^p \ln(x_i)$ instead. The extreme points of the constrained problem satisfy the Lagrange Multipliers equation and $f(x_1, \ldots, x_p) = 1$. There is $\lambda \in \mathbf{R}$ so that

$$\nabla \ln \phi(x_1, \ldots, x_p) = \lambda \nabla f(x_1, \ldots, x_p) \implies \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_p}\right) = \lambda(1, 1, \ldots, 1).$$

In other words $x_1 = \frac{1}{\lambda}$ for all i. Putting this into the constraint, $1 = \sum_{i=1}^p x_i = \sum_{i=1}^p \frac{1}{\lambda} = \frac{p}{\lambda}$ so $\lambda = \frac{1}{p}$ at the critical point and there $\phi(x_1, \ldots, x_p) = \left(\frac{1}{p}\right)^p$. Observe that $0 < x_i = 1 - \sum_{j \neq i} x_j < 1$ so $S = \{\mathbf{x} \in U : f(\mathbf{x}) = 1\}$ is bounded. As ϕ is uniformly continuous, $\phi(S)$ is bounded also. $x_i > 0$ implies $\phi(S) \subseteq (0, \infty)$. As $x_i \to 0$ so $\phi \to 0$. Thus $\phi(S) = (0, b]$ since S is connected. Also b is a max of ϕ, which is taken at an inner point of S so must be the critical point we found because there was only one critical point. Hence the critical point is a maximum. Note that from this problem one can deduce the Arithmetic-Geometric Mean Inequality: $\left(\prod_{i=1}^p |x_i|\right)^{\frac{1}{p}} \leq \frac{1}{p}\sum_{i=1}^p |x_i|$.

(5A.) Statement. *Suppose* $f : \mathbf{R}^p \to \mathbf{R}$ *is differentiable at* $\mathbf{a} \in \mathbf{R}^p$. *Then the partial derivatives* $\frac{\partial f}{\partial x_i}$ *exists at* $\mathbf{a}$.

TRUE. By specializing to the limit $\mathbf{h} = (0, 0, \ldots, 0, h_i, 0, \ldots, 0)$, the limit (1a.) says $\frac{\partial f}{\partial x_i}(\mathbf{a}) = m_{1,i}$, so that the partial derivative is the corresponding entry in the Jacobian matrix $M = df(\mathbf{a})$. This follows from

$$\lim_{hi \to 0} \frac{f(a_1, \ldots, a_{i-1} a_i + h_i, a_{i+1}, \ldots, a_p)) - f(a_1, \ldots, a_p) - m_{1,i}h_i}{h_i} \cdot \frac{h_i}{\|\mathbf{h}\|} = 0.$$

(5b.) Statement. *Suppose* $f : \mathbf{R}^p \to \mathbf{R}$ *is twice continuously differentiable. Suppose* f *has a local minimum at* $\mathbf{a} \in \mathbf{R}^p$. *Then* $d^2f(\mathbf{a})$ *is positive definite.*

FALSE. $f(x, y) = x^4 + y^4$ has a global minimum at $(0, 0)$ because $x^4 \geq 0$ and $y^4 \geq 0$ but the Hessian matrix is dead zero at the minimum $(x, y) = (0, 0)$

$$d^2f((x,y)) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} \implies d^2f((0,0)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is not positive definite because for any $\mathbf{h} \neq 0$, $4d^2f((0,0))(\mathbf{h})^2 = 0$ and not positive as it should be for a positive definite matrix.

(5c.) Statement. *Let* $f(x, y) = \frac{xy}{x-y}$. *Then there is* $c \in (1, 6)$ *such that* $f(6, 4) = f(1, 4) + df(c, 4)\binom{5}{0}$.

FALSE. The conditions for the mean value theorem fail at the point $(4, 4) \in [(1, 4), (6, 4)]$ in the line segment. At that point f blows up. You can see it another way. $\frac{\partial f}{\partial x} = -\frac{y^2}{(x-y)^2}$ which is negative. However, if there were c, then

$0 < 12 - \left(-\frac{4}{3}\right) = f(6, 4) - f(1, 4) = df(c, 4)\binom{5}{0} = 5\frac{\partial f}{\partial x}(c, 4) + 0 \cdot \frac{\partial f}{\partial y}(c, 4) = -\frac{80}{(c-4)^2} < 0$
which is a contradiction.