Math 3220 § 2. **Second Midterm Exam** Name:____________________

Treibergs February 27, 2013

(1.) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $(a, b) \in \mathbb{R}^2$ a point. State the definition: f is a continuous at (a, b). Determine whether f is continuous at $(0, 0)$ and prove your answer, where*

$$f(x,y) = \begin{cases} \dfrac{(x-y)^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Definition. $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(a, b) \in \mathbb{R}^2$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|f(x,y) - f(a,b)\| < \varepsilon \qquad \text{whenever } (x,y) \in \mathbb{R}^2 \text{ and } \|(x,y) - (a,b)\| < \delta.$$

We observe that the function is cubic over quadratic, so it should tend to zero at the origin. Indeed, since

$$|x - y| \le |x| + |y| \le 2\sqrt{x^2 + y^2} = 2\|(x,y)\|$$

we have for $(x, y) \neq (0, 0)$,

$$|f(x,y) - f(0,0)| = \left| \frac{(x-y)^3}{x^2+y^2} \right| \le \frac{8\|(x,y)\|^3}{\|(x,y)\|^2} \le 8\|(x,y)\|.$$

Thus, if we choose $\varepsilon > 0$ and take $\delta = \varepsilon/8$, then for any $(x, y) \in \mathbb{R}^2$ such that $\|(x, y) - (0, 0)\| < \delta$ we have either $(x, y) = (0, 0)$ in which case $|f(x, y) - f(0, 0)| = 0 < \epsilon$ or $(x, y) \neq (0, 0)$ in which case

$$|f(x,y) - f(0,0)| \le 8\,\|(x,y)\| = 8\,\|(x,y) - (0,0)\| < 8\delta = \varepsilon.$$

Hence, f is continuous at $(0, 0)$. □

(2.) *Let $f : \mathbb{R}^p \to \mathbb{R}$ be continuous and $a < b$. Using just the definitions of continuity and openness, show that E is open, where*

$$E = \{x \in \mathbb{R}^p : a < f(x) < b\}.$$

To show E is open we must show that for every $x \in E$ there is a $\delta > 0$ such that the whole open ball $B_\delta(x) \subset E$. Choose $x \in E$. Thus $a < f(x) < b$. Since the interval is open, there is an $\varepsilon > 0$ such that $\big(f(x) - \varepsilon, f(x) + \varepsilon\big) \subset (a, b)$. By continuity, there is a $\delta > 0$ such that $|f(z) - f(x)| < \varepsilon$ whenever $z \in \mathbb{R}^p$ and $\|z - x\| < \delta$. But this implies that if $z \in B_\delta(x)$, then

$$f(z) \in \big(f(x) - \varepsilon, f(x) + \varepsilon\big) \subset (a,b).$$

In other words, $B_\delta(x) \subset E$. Hence, E is open. □

(3.) *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

1. **Statement** *Suppose $K \subset \mathbb{R}^q$ is compact and $f : \mathbb{R}^p \to \mathbb{R}^q$ is continuous. Then $f^{-1}(K)$ is compact.*

   FALSE. For example, let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be given by $f(x) = e^x$. Then $K = [-1, 1]$ is compact but $f^{-1}(K) = (-\infty, 0]$ is not compact (because it is unbounded.)

2. **Statement.** *Let $E \subset \mathbb{R}^2$ be a connected set in the plane. Then the boundary ∂E is connected.*

   FALSE. For example let $E = B_2(0) - \overline{B_1(0)}$ be the open annulus whose boundary $\partial E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$ is two disconnected circles.

3. **Statement.** *Suppose that $K \subset \mathbb{R}^p$ is compact and $f_n, f : K \to \mathbb{R}^q$. If the $\{f_n\}$ converges to f pointwise on K then it also converges uniformly on K.*

   FALSE. Let $K = [0, 1] \subset \mathbb{R}^1$ and $f_n : K \to \mathbb{R}^1$ be given by $f_n(x) = x^n$. Then $\{f_n\}$ converges pointwise to $f(x) = 0$ if $0 \le x < 1$ and $f(1) = 1$. The convergence is not uniform because if it were, then the uniform limit f would have to be continuous because all f_n are.

(4.) *Let $D \subset \mathbb{R}^p$ and $f, f_n : D \to \mathbb{R}^q$ be transformations. State the definition: $\{f_n\}$ converges uniformly to f on D. Suppose that all of the transformations satisfy the same Lipschitz condition: there is a constant $c \in \mathbb{R}$ such that*

$$\|f_n(x) - f_n(y)\| \le c\|x - y\| \qquad \textit{whenever } n \in \mathbb{N} \textit{ and } x, y \in D.$$

Show that if $f_n \to f$ uniformly on D, then f also satisfies the Lipschitz condition.

Definition. Let $f, f_n : D \to \mathbb{R}^q$. $\{f_n\}$ converges to f uniformly if for every $\varepsilon > 0$ there is a $N \in \mathbb{R}$ such that

$$\|f_n(x) - f(x)\| < \varepsilon \qquad \text{whenever } x \in D \text{ and } n > N.$$

Let f be the uniform limit of $\{f_n\}$ on D. We show that f satisfies the Lipschitz condition with the same c. To estimate the difference, choose any $x, y \in D$. Choose $\varepsilon > 0$. By uniform convergence, there is a $N \in \mathbb{R}$ such that $\|f_n(z) - f(z)\| < \varepsilon$ whenever $z \in D$ and $n > N$. Take any $m \in \mathbb{N}$ such that $m > N$. By the triangle inequality, the Lipschitz condition for f_m and taking $z = x$ and $z = y$ we obtain

$$\begin{aligned} \|f(x) - f(y)\| &= \|f(x) - f_m(x) + f_m(x) - f_m(y) + f_m(y) - f(y)\| \\ &\le \|f(x) - f_m(x)\| + \|f_m(x) - f_m(y)\| + \|f_m(y) - f(y)\| \\ &< \varepsilon + c\|x - y\| + \varepsilon. \end{aligned}$$

Since ε may be any positive number, it follows that

$$\|f(x) - f(y)\| \le c\|x - y\|.$$

Since x, y were arbitrary, this inequality holds whenever $x, y \in D$. □

(5.) *Let* $f : \mathbb{R}^p \to \mathbb{R}$ *be a real function. State the definition:* f *has a partial derivative* $\dfrac{\partial f}{\partial x_j}(a)$ *with respect to the* j*th variable at* $a \in \mathbb{R}^p$*. Determine whether* f *has a partial derivative with respect to* y *at* $(0,0)$*, where*

$$f(x,y) = \begin{cases} \dfrac{x^4+y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Definition. $f : \mathbb{R}^p \to \mathbb{R}$ has a partial derivative with respect to the jth variable at $a \in \mathbb{R}^p$ if the following limit exists

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{j-1}, a_j + h, a_{j+1}, \ldots, a_n) - f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)}{h}.$$

For the given function, if $y \neq 0$ then

$$f(0,y) = \frac{0^4 + y^3}{0^2 + y^2} = y,$$

and if $y = 0$, $f(0,0) = 0$. In both cases $f(0,y) = y$. It follows that

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

Alternatively, the partial derivative is defined if the one variable derivative exists:

$$\frac{\partial f}{\partial x_j}(a) = \left.\frac{d}{dt}\right|_{t=a_j} f(a_1, \ldots, a_{j-1}, t, a_{j+1}, \ldots, a_n).$$

So for this function

$$\frac{\partial f}{\partial y}(0,0) = \left.\frac{d}{dt}\right|_{t=0} f(0,t) = \left.\frac{d}{dt}\right|_{t=0} t = 1.$$