Math 3220 § 2. **Sample Problems for Third Midterm** Name:______________
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Questions 1–4 appeared in my Fall 2000 and Fall 2001 Math 3220 exams

(1.)*Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable at $a \in \mathbf{R}^n$.*

(a.) Let $g : \mathbf{R}^n \to \mathbf{R}$ be defined by $g(x) = x \bullet f(x)$, (dot product.) Find the total derivative (differential) $Dg(a)(h)$ where $h \in \mathbf{R}^n$.

(b.) Without using the product theorem, prove your answer.

(2.) For each part, determine whether the statement is TRUE or FALSE. If the statement is true, give a justification. If the statement is false, give a counterexample. You may use theorems.

(a.) **Statement.** Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous. Suppose that for all $(x, y) \in \mathbf{R}^2$ both

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \qquad \text{and} \qquad \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

exist. Then f is differentiable at $(1, 2)$.

(b.) **Statement.** Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$ function for which the third partial derivatives $f_{xxy}(x, y)$ exist for all $(x, y) \in \mathbf{R}^2$ such that $f_{xxy}(x, y)$ is continuous at $(0, 0)$. Then $f_{xyx}(0, 0)$ and $f_{yxx}(0, 0)$ exist and are equal: $f_{xxy}(0, 0) = f_{xyx}(0, 0) = f_{yxx}(0, 0)$.

(3.) Suppose $\mathbf{f}, \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$. Assume that $\mathbf{g}$ is differentiable at $\mathbf{x}_0 \in \mathbf{R}^2$ and that for some $\alpha > 1$ and $M < \infty$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\| \le M\|\mathbf{x} - \mathbf{x}_0\|^\alpha$$

for all $\mathbf{x} \in \mathbf{R}^2$. Show that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ and that $D\mathbf{f}(\mathbf{x}_0) = D\mathbf{g}(\mathbf{x}_0)$.

(4.) Let $F : \mathbf{R}^5 \to \mathbf{R}^2$ be given by $F = (f_1, f_2)$ where

$$\begin{aligned} f_1(v, w, x, y, z) &= v + w^2 + x + y, \\ f_2(v, w, x, y, z) &= vy + wz. \end{aligned}$$

Suppose that there is an open neighborhood $U \subseteq \mathbf{R}^3$ containing the point $(3, 4, 5)$ and $\mathcal{C}^1$ functions $G : U \to \mathbf{R}^2$ where $G = (g_1, g_2)$ so that for all $(x, y, z) \in U$,

$$\begin{aligned} g_1(3,4,5) &= 1, & f_1(g_1(x,y,z), g_2(x,y,z), x, y, z) &= 12, \\ g_2(3,4,5) &= 2; & f_2(g_1(x,y,z), g_2(x,y,z), x, y, z) &= 14. \end{aligned}$$

What is the differential $DG(3, 4, 5)(h, k, \ell)$?

More problems.

(E1.) Suppose $f : \mathbf{R}^3 \to \mathbf{R}^2$ is given by $f(x, y, z) = (xy + x^2z^3, x^4 + y + y^5z^6)$. Is f differentiable on $\mathbf{R}^3$? If so, find the differential $df(x, y, z)(h, k, \ell)$.

(E2.) Find the extrema of $\phi(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x - y = 1$ and $y^2 - z^2 = 1$.

(E3.) Suppose that r and α are positive, $E \subseteq \mathbf{R}^n$ is a convex set such that $\bar{E} \subseteq B_r(0)$, and that there exists a sequence $\mathbf{x}_k \in E$ such that $\mathbf{x}_k \to 0$ as $k \to \infty$. If $f : B_r(0) \to \mathbf{R}$ is continuously differentiable and $|f(\mathbf{x})| \le \|\mathbf{x}\|^\alpha$ for all $\mathbf{x} \in E$, prove that there is an $M < \infty$ such that $|f(\mathbf{x})| \le M\|\mathbf{x}\|$ for $\mathbf{x} \in E$.

(E4.) Show that $f(x,y)$ has partial derivatives for all $(x,y) \in \mathbf{R}^2$ but f is not differentiable at $(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3 + 4xy^2}{x^2 + 2y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(E5.) Suppose that $f : \mathbf{R}^p \to \mathbf{R}$ is differentiable at $\mathbf{a} \in \mathbf{R}^p$ and $df(\mathbf{a}) \neq 0$. Show that $\nabla f(\mathbf{a})$ points in the direction of fastest increase (that has the largest directional derivative.)

(E6.) Let $S \subseteq \mathbf{R}^4$ be a locally parameterized $\mathcal{C}^1$ 2-dimensional surface near $\mathbf{b} = (0,2,0,2)$, where

$$S = \{(x,y,z,w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 = 8,\ x + y + z - w = 0\}.$$

Show that the tangent plane to S at $\mathbf{b}$ is $\{(s, 2 - \frac{s}{2} - \frac{t}{2}, t, 2 + \frac{s}{2} + \frac{t}{2}) : s,t \in \mathbf{R}\}$.

(E7.) Suppose that $f : \mathbf{R}^p \to \mathbf{R}$ is $\mathcal{C}^3$ and some point $\mathbf{a} \in \mathbf{R}^p$ is critical $df(\mathbf{a}) = 0$ and $d^2 f(\mathbf{a})$ has both a positive and a negative eigenvalue. Show that the critical point $\mathbf{a}$ is a saddle point: for every $\delta > 0$ there are points $\mathbf{x}, \mathbf{y} \in B_\delta(\mathbf{a})$ so that $f(\mathbf{x}) < f(\mathbf{a}) < f(\mathbf{y})$.

(E7*.) (Variation of above.) Suppose that $f : \mathbf{R}^p \to \mathbf{R}$ is $\mathcal{C}^1$ function such that some point $\mathbf{a} \in \mathbf{R}^p$ is is a critical point: $df(\mathbf{a}) = 0$. Assume that the second partial derivatives exist in a neighborhood of $\mathbf{a}$, are continuous at $\mathbf{a}$ and that the Hessian $d^2 f(\mathbf{a})$ has both a positive and a negative eigenvalue. Show that the critical point $\mathbf{a}$ is a saddle point: for every $\delta > 0$ there are points $\mathbf{x}, \mathbf{y} \in B_\delta(\mathbf{a})$ so that $f(\mathbf{x}) < f(\mathbf{a}) < f(\mathbf{y})$.

(E8.) Let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $F(x,y,z) = (x^2 + y^2, xz, y^3 - z^3)$.

Assume that there is an open set U about $P_0 = (3,1,2)$ and that $V = F(U)$ is an open set about $Q_0 = F(P_0)$ on which F has an inverse function $F^{-1} \in \mathcal{C}^1(V,U)$. Find $D[F^{-1}](Q_0)$.

(E9.) Suppose $G : \mathbf{R}^5 \to \mathbf{R}^3$ is given by $G(p,q,x,y,z) = (px + y^2, q^2 z, py - qz + x)$. Assume that there is an open set U around $(3,2)$ and a $\mathcal{C}^1$ function $H : U \to \mathbf{R}^3$ so that $H(3,2) = (1,5,4)$ and for all $(p,q) \in U$ we have $G(p,q,H(p,q)) = (28,16,8)$. Find $DH(3,2)$.

Solutions.

(1.)*Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable at $a \in \mathbf{R}^n$. Let $g : \mathbf{R}^n \to \mathbf{R}$ be defined by $g(x) = x \bullet f(x)$, (dot product.) Find the total derivative (differential) $Dg(a)(h)$ where $h \in \mathbf{R}^n$. Without using the product theorem, prove your answer.*

The product rule gives $Dg(x)(h) = D(x \bullet f(x))(h) = h \bullet g(x) + x \bullet Df(x)(h)$. This is the differential because

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|} = \\
&= \lim_{h\to 0} \frac{\|(x+h) \bullet f(x+h) - x \bullet f(x) - h \bullet f(x) - x \bullet Df(x)(h)\|}{\|h\|} \\
&= \lim_{h\to 0} \frac{\|x \bullet (f(x+h) - f(x) - Df(x)(h)) + h \bullet (f(x+h) - f(x))\|}{\|h\|} \\
&\le \lim_{h\to 0} \left\{ \|x\| \frac{\|f(x+h) - f(x) - Df(x)(h)\|}{\|h\|} + \frac{\|h\|}{\|h\|} \|f(x+h) - f(x)\| \right\} \\
&= \|x\| \cdot 0 + 1 \cdot 0 = 0,
\end{aligned}$$

where we have used the Schwarz Inequality.

(2a.) **Statement.** *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous. Suppose that for all $(x,y) \in \mathbf{R}^2$ both*

$$\lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad \text{and} \qquad \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

exist. Then f is differentiable at $(1,2)$.

FALSE! The two limits are nothing more than $f_x(x,y)$ and $f_y(x,y)$, the partial derivatives. There are functions where the partial derivatives exist at all points, but the function is not differentiable. (If it were known that f_x and f_y are continuous at some point $\mathbf{a} \in \mathbf{R}^2$, then our theorem says that the function would be differentiable at $\mathbf{a}$.) An example of such a function is

$$f(x,y) = \begin{cases} \frac{(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}, & \text{if } (x,y) \neq (1,2); \\ 0. & \text{if } (x,y) = (1,2). \end{cases}$$

Away from $(1,2)$, the denominator avoids zero, so the partial derivatives exist and are continuous, hence f is differentiable. Also, observe that $f(1,y) = f(x,2) = 0$ for all x, y. Hence $f_y(1,y) = f_x(x,2) = 0$ so the partial derivatives exist at $(1,2)$. If the function were differentiable at $(1,2)$, then the vanishing of the partial derivatives implies that the differential would have to be $T(h,k) = 0$ all h, k. But the limit

$$\lim_{(h,k)\to(1,2)} \frac{\|f(1+h,2+k) - f(1,2) - T(h,k)\|}{\|(h,k)\|} = \lim_{(h,k)\to(1,2)} \frac{\|f(1+h,2+k)\|}{\|(h,k)\|}$$

does not exist. To see this, consider the first approach $(h,k) = (t,0)$ as $t \to 0$. The numerator vanishes so along this approach the limit would be zero. Then consider the second approach $(h,k) = (t,t)$ for $t > 0$. Then $f(1+t,2+t) = t/2$ and $\|(t,t)\| = \sqrt{2}|t|$. Then the difference quotient tends to $1/\sqrt{2}$. Since the two approaches have different limits, there is no two dimensional limit: the function is not differentiable at $(1,2)$.

(2b.) **Statement.** *Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a $\mathcal{C}^2$ function for which the third partial derivatives $f_{xxy}(x,y)$ exist for all $(x,y) \in \mathbf{R}^2$ such that $f_{xxy}(x,y)$ is continuous at $(0,0)$. Then $f_{xyx}(0,0)$ and $f_{yxx}(0,0)$ exist and are equal $f_{xxy}(0,0) = f_{xyx}(0,0) = f_{yxx}(0,0)$.*

TRUE! This is just an application of the equality of cross partials theorem to f_x and f_y which are $\mathcal{C}^1$ by assumption, since f is $\mathcal{C}^2$. We are given that $(f_x)_{xy}$ exists and is continuous at $(0,0)$. But, this is sufficient to be able to assert the existence of the other mixed partial derivative, and that it is equal to the first at the point $(f_x)_{xy}(0,0) = (f_x)_{yx}(0,0)$. But since $f \in \mathcal{C}^2(\mathbf{R}^2)$, we also have that $f_{xy} = f_{yx}$ for all of $\mathbf{R}^2$. Hence, all third derivatives exist and are equal $f_{xxy}(0,0) = (f_x)_{xy}(0,0) = (f_x)_{yx}(0,0) = (f_{xy})_x(0,0) = (f_{yx})_x(0,0) = f_{yxx}(0,0)$.

(3.) *Suppose $\mathbf{f}, \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^2$. Assume that $\mathbf{g}$ is differentiable at $\mathbf{x}_0 \in \mathbf{R}^2$ and that for some $\alpha > 1$ and $M < \infty$ we have*

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\| \le M\|\mathbf{x} - \mathbf{x}_0\|^\alpha$$

for all $\mathbf{x} \in \mathbf{R}^2$. Show that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$ and that $D\mathbf{f}(\mathbf{x}_0) = D\mathbf{g}(\mathbf{x}_0)$.

It suffices to show that the difference quotient of f limits to zero with differential $dg(x_0)$. Now, by adding and subtracting, using the triangle inequality, the hypothesis and $\alpha > 1$,

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - Dg(x_0)(h)\|}{\|h\|} = \\
&= \lim_{h\to 0} \frac{\|f(x_0+h) - g(x_0+h) - f(x_0) + g(x_0) + g(x_0+h) - g(x_0) - Dg(x_0)(h)\|}{\|h\|} \\
&\le \lim_{h\to 0} \frac{\|f(x_0+h) - g(x_0+h)\| + \|g(x_0) - f(x_0)\| + \|g(x_0+h) - g(x_0) - Dg(x_0)(h)\|}{\|h\|} \\
&\le \lim_{h\to 0} \left\{ \frac{\|h\|^\alpha}{\|h\|} + \frac{\|0\|^\alpha}{\|h\|} + \frac{\|g(x_0+h) - g(x_0) - Dg(x_0)(h)\|}{\|h\|} \right\} \\
&= \lim_{h\to 0} \|h\|^{\alpha-1} + 0 + \lim_{h\to 0} \frac{\|g(x_0+h) - g(x_0) - Dg(x_0)(h)\|}{\|h\|} = 0 + 0 + 0.
\end{aligned}$$

(4*.) *(Slight generalization.) Let* $F : \mathbf{R}^5 \to \mathbf{R}^2$ *be given by* $F = (f_1, f_2)$ *where* $f_1(v, w, x, y, z) = v + w^2 + x + y$, $f_2(v, w, x, y, z) = vy + wz$. *Show that there is a neighborhood* $U \subseteq \mathbf{R}^3$ *containing the point* $(3, 4, 5)$ *and* $\mathcal{C}^1$ *functions* $G : U \to \mathbf{R}^2$ *where* $G = (g_1, g_2)$ *so that* $g_1(3, 4, 5) = 1$, $g_2(3, 4, 5) = 2$ *and for all* $(x, y, z) \in U$, $f_1(g_1(x, y, z), g_2(x, y, z), x, y, z) = 12$, $f_2(g_1(x, y, z), g_2(x, y, z), x, y, z) = 14$. *Find the total derivative* $DG(3, 4, 5)(h, j, k)$.

You were given the first conclusion in problem (4.) which can be answered knowing the chain rule. (4*) is an application of the Implicit Function Theorem. (This theorem says that if there is enough differentiability, and if the problem can be solved for the linear approximations given by the differential, then, at least in a small neighborhood, the nonlinear problem can be solved as well.) The function $F : \mathbf{R}^{2+3} \to \mathbf{R}^2$ is $\mathcal{C}^1$ on $\mathbf{R}^5$ such that $F(1, 2, 3, 4, 5) = (12, 14) = \mathbf{c}$. To solve for v and w in terms of (x, y, z) we need to be able to solve the linearization. If we put $\mathbf{u} = (v, w)$ and $\mathbf{x} = (x, y, z)$, we are looking for $G : \mathbf{R}^3 \to \mathbf{R}^2$ so that $F(G(\mathbf{x}), \mathbf{x}) = \mathbf{c}$ and $G(3, 4, 5) = (1, 2)$. Taking $D_\mathbf{x}$ gives $D_\mathbf{u}F(G(\mathbf{x}), \mathbf{x}) \circ DG(\mathbf{x}) + D_\mathbf{x}F(G(\mathbf{x}), \mathbf{x}) = 0$ which says that we may solve for the differential $DG(\mathbf{x})$ whenever $D_\mathbf{u}F(G(\mathbf{x}), \mathbf{x})$ is invertible and then $DG(\mathbf{x}) = -[D_\mathbf{u}F(G(\mathbf{x}), \mathbf{x})]^{-1} \circ D_\mathbf{x}F(G(\mathbf{x}), \mathbf{x})$. At the center point $(3, 4, 5)$, the matrix of the transformation is

$$D_\mathbf{u}F(G(3,4,5),(3,4,5)) = \begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{pmatrix}\Bigg|_{\mathbf{x}=(3,4,5)} = \begin{pmatrix} 1 & 2w \\ y & z \end{pmatrix}\Bigg|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)} = \begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix}$$

which is invertible. Hence there is an open set $U \subseteq \mathbf{R}^3$ such that $(3, 4, 5) \in U$ and a $\mathcal{C}^1$ function $G : U \to \mathbf{R}^2$ so that $G(2, 3, 4) = (1, 2)$ and $F(G(\mathbf{x}), \mathbf{x}) = \mathbf{c}$ for all $\mathbf{x} \in U$. Thus, we have checked the differentiability and the solubility of the linearized problem is satisfied. The IFT gives the existence of a $G \in \mathcal{C}^1(U, \mathbf{R}^2)$ so that $G(2, 3, 4) = (1, 2)$ and $F(G(\mathbf{x}), \mathbf{x}) = \mathbf{c}$ for all $\mathbf{x} \in U$. In this problem, ypu were given this as a hypothesis. Then take the total derivative of $F(G(\mathbf{x}), \mathbf{x}) = \mathbf{c}$ with respect to $\mathbf{x}$ and solve for $DG(\mathbf{x})$ at the given point, as above.) By the formula for the differential

$$DG(3,4,5)\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = -[D_\mathbf{u}F(G(3,4,5),(3,4,5))]^{-1} \circ D_\mathbf{x}F(G(3,4,5),(3,4,5))\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= -\left[\begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{pmatrix}\Bigg|_{\mathbf{x}=(3,4,5)}\right]^{-1} \left\{\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}\Bigg|_{\mathbf{x}=(3,4,5)}\right\}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= -\left[\begin{pmatrix} 1 & 2w \\ y & z \end{pmatrix}\Bigg|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)}\right]^{-1} \left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & v & w \end{pmatrix}\Bigg|_{(\mathbf{u},\mathbf{x})=(1,2,3,4,5)}\right\}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$DG(3,4,5)\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = -\begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5 & -4 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} h \\ k \\ \ell \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} -5 & -1 & 8 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -5h - k + 8\ell \\ 4h + 3k - 2\ell \end{pmatrix}. \quad \square$$

(E1.) *Suppose $f : \mathbf{R}^3 \to \mathbf{R}^2$ is given by $f(x,y,z) = (xy + x^2z^3, x^4 + y + y^5z^6)$. Is f differentiable on $\mathbf{R}^3$? If so, find the differential $df(x,y,z)(h,k,\ell)$.*

YES! The partial derivatives are

$$\frac{\partial f}{\partial x}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y + 2xz^3 \\ 4x^3 \end{pmatrix}, \qquad \frac{\partial f}{\partial y}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 1 + 5y^4z^6 \end{pmatrix}, \qquad \frac{\partial f}{\partial z}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x^2z^2 \\ 6y^5z^5 \end{pmatrix}$$

Since f is a polynomial function, its first partial derivatives exist at all points and are polynomial functions. But by the theorem giving conditions for differentiability, since the partial derivatives are continuous at all points, the function is differentiable at all points. The differential is given by the 2×3 Jacobian matrix

$$df\begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix} = \begin{pmatrix} y + 2xz^3 & x & 3x^2z^2 \\ 4x^3 & 1 + 5y^4z^6 & 6y^5z^5 \end{pmatrix}\begin{pmatrix} h \\ k \\ \ell \end{pmatrix}.$$

(E2.) *Find the extrema of $\phi(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints $x - y = 1$ and $y^2 - z^2 = 1$.*

Let $g(x,y,z) = x - y - 1$ and $h(x,y,z) = y^2 - z^2 - 1$. Then using Lagrange Multipliers, the extrema occur as solutions (x, y, z, λ, μ) of the system $g = 0$, $h = 0$, $\nabla\phi = \lambda\nabla g + \mu\nabla h$. Hence

$$(2x, 2y, 2z) = \lambda(1, -1, 0) + \mu(0, 2y, -2z).$$

Thus $2x = \lambda$, $2y = -\lambda + 2\mu y$ and $z = -\mu z$. From the last equation, either $z = 0$ or $\mu = -1$.

If $\mu = -1$ then $\lambda = -4y$. Since $2x = \lambda$ we get $x + 2y = 0$. Now $g = 0$ implies $y = -1/3$. However, $h = 0$ implies $0 = 1/9 - z^2 - 1$ which is a contradiction.

If $z = 0$ then $h = 0$ implies $y = \pm 1$. Since $g = 0$ we have $x = 2$ when $y = 1$ and $x = 0$ when $y = -1$. Thus the only candidates for extrema subject to the constraints $g = h = 0$ are $\phi(2,1,0) = 5$ and $\phi(0,-1,0) = 1$. To see whether these are maxima or minima, consider the geometric interpretation. The problem asks to find the closest or farthest points of the solution set $g = h = 0$ to the origin. However $h = 0$ corresponds to a hyperbolic cylinder (union of lines parallel to the x-axis that pass through a hyperbola in the y-z plane.) The constraint $g = 0$ is a plane parallel to the z-axis that crosses the hyperboloid. Thus the constraint set consists of two hyperbolas in the $g = 0$ plane. The level curves on ϕ are circles in this plane. As there are only two candidates for extrema, these occur when the circles meet (are tangent to) the two arcs of the hyperbola. Larger circles cross the arcs. This means that $(0,-1,0)$ is a minimum and $(2,1,0)$ is a local but not global minimum, and there is no maximum since one can attain arbitrarily large ϕ on the constraint set.

(E3.) *Suppose that r and α are positive, $E \subseteq \mathbf{R}^n$ is a convex set such that $\bar{E} \subseteq B_r(0)$, and that there exists a sequence $\mathbf{x}_k \in E$ such that $\mathbf{x}_k \to 0$ as $k \to \infty$. If $f : B_r(0) \to \mathbf{R}$ is continuously differentiable and $|f(\mathbf{x})| \le \|\mathbf{x}\|^\alpha$ for all $\mathbf{x} \in E$, prove that there is an $M < \infty$ such that $|f(\mathbf{x})| \le M\|\mathbf{x}\|$ for $\mathbf{x} \in E$.*

Since the partial derivatives are continuous, the function $\mathbf{x} \mapsto \|df(\mathbf{x})\|$ is continuous. As $\bar{E}$ is closed and bounded, it is compact so the continuous function attains its maximum $M = \sup\{\|df(\mathbf{x})\| : \mathbf{x} \in \bar{E}\} < \infty$. Since E is convex, for any pair of points $\mathbf{x}, \mathbf{x}_k \in E$ the line segment $[\mathbf{x}_k, \mathbf{x}]$ is in E. Since f is differentiable in a neighborhood of $[\mathbf{x}_k, \mathbf{x}]$, we may apply the Mean Value Theorem. There is a $\mathbf{c} \in [\mathbf{x}_k, \mathbf{x}]$ so that

$$f(\mathbf{x}) = f(\mathbf{x}_k) + df(\mathbf{c})(\mathbf{x} - \mathbf{x}_k)$$

Estimating using triangle and Schwarz inequalities, and the hypothesis

$$\begin{aligned} |f(\mathbf{x})| &\le |f(\mathbf{x}_k)| + \|df(\mathbf{c})\| \|\mathbf{x} - \mathbf{x}_k\| \\ &\le \|\mathbf{x}_k\|^\alpha + M\|\mathbf{x} - \mathbf{x}_k\|. \end{aligned}$$

Now since $\alpha > 0$, by passing to the limit as $k \to \infty$ we obtain the estimate

$$|f(\mathbf{x})| \le 0 + M\|\mathbf{x}\|.$$

(E4.) *Theorem. $f(x,y)$ has partial derivatives for all $(x,y) \in \mathbf{R}^2$ but f is not differentiable at $(0,0)$, where*

$$f(x,y) = \begin{cases} \dfrac{x^3 + 2y^3 + 4xy^2}{x^2 + 2y^2}, & \text{if } (x,y) \ne (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Since f is a rational function whose denominator is nonzero away from $(0,0)$, the partial derivatives exist and are rational functions there. At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{2h^3}{2h^2} - 0}{h} = 1$$

so both partial derivatives exist at $(0,0)$ as well. Therefore, supposing that f were differentiable at $(0,0)$, its differential would be $df(0,0) = (1,1)$. Now lets check if this differential well-approximates f near zero.

$$\begin{aligned} \lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - df(0,0)\binom{h}{k}}{\|(h,k) - (0,0)\|} &= \lim_{(h,k)\to(0,0)} \frac{\frac{h^3+2k^3+4hk^2}{h^2+2k^2} - 0 - (h+k)}{\sqrt{h^2+k^2}} \\ &= \lim_{(h,k)\to(0,0)} \frac{2hk^2 - h^2k}{(h^2+2k^2)\sqrt{h^2+k^2}}. \end{aligned}$$

Along the path $(h,k) = (t,0)$ the limit is zero. However, taking the path $(h,k) = (t,t)$, the limit is $1/(3\sqrt{2})$. Because the limits along two approach paths disagree, there is no two dimensional limit. The function is not well approximated by the only possible affine function, hence it is not differentiable at $(0,0)$.

(E5.) *Theorem. Suppose that* $f : \mathbf{R}^p \to \mathbf{R}$ *is differentiable at* $\mathbf{a} \in \mathbf{R}^p$ *and* $df(\mathbf{a}) \neq 0$. *Then* $\nabla f(\mathbf{a})$ *points in the direction of fastest increase (that has the largest directional derivative.)*

Let $\mathbf{u}$ be a unit vector. Then the directional derivative in the $\mathbf{u}$ direction is $D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$. By the Schwarz inequality $|D_{\mathbf{u}} f(\mathbf{a})| \le \|\nabla f(\mathbf{a})\| \|\mathbf{u}\| = \|\nabla f(\mathbf{a})\|$ with equality only if $\mathbf{u} = \pm \nabla f(\mathbf{a}) / \|\nabla f(\mathbf{a})\|$. Thus $D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{a})\|$ if and only if $\mathbf{u} = \nabla f(\mathbf{a}) / \|\nabla f(\mathbf{a})\|$, which is in the gradient direction.

(E6.) *Theorem.* $S \subseteq \mathbf{R}^4$ *is a locally parameterized 2-dimensional surface near* $\mathbf{b} = (0,2,0,2)$, *where*

$$S = \{(x,y,z,w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 = 8,\ x + y + z - w = 0\}.$$

The tangent plane to S *at* $\mathbf{b}$ *is* $\{(s, 2 - \frac{s}{2} - \frac{t}{2}, t, 2 + \frac{s}{2} + \frac{t}{2}) : s,t \in \mathbf{R}\}$.

The surface is the intersection of a 3-plane through the origin and the 3-sphere about the origin of radius $2\sqrt{2}$ which is a great 2-sphere. Let's write the surface near $\mathbf{b}$ as a graph over the (x,z)-plane, for $x^2 + z^2 < 1$. Note that the x and z coordinates of $\mathbf{b}$ are in this open neighborhood, and that the 3-plane is not perpendicular to the $x - z$-plane. Solving the second equation gives $y = w - x - z$. Substituting $x = u$ and $z = v$ into the first and solving for w gives $G : U \to \mathbf{R}^4$ where we take the "+" square root because $G(0,0) = (0,2,0,2)$ and $U = B_1((0,0))$:

$$G(u,v) = \left(u, \frac{-u - v + \sqrt{16 - 3u^2 - 2uv - 3v^2}}{2}, v, \frac{u + v + \sqrt{16 - 3u^2 - 2uv - 3v^2}}{2} \right)$$

$G \in C^1(U, \mathbf{R}^4)$ because $16 - 3(u^2 + v^2) - 2uv > 16 - 3 \cdot 1 - 1 = 12$ since $2uv \le u^2 + v^2 < 1$. One checks that $G(u,v) \in S$. Let $V = \{(x,y,z,w) : x^2 + z^2 < 1, y > 0, w > 0\}$. One checks that $\mathbf{b} \in S \cap V = G(U)$. We need that $G : U \to S \cap V$ is one-to-one. But if $(u_i, v_i) \in U$ and $G((u_1,v_1)) = G((u_2,v_2))$ then x- and z-coordinate functions give $u_1 = u_2$ and $v_1 = v_2$ so G is one-to-one. Finally we check that G is two dimensional. $dG((u,v))$ is a 4×2 matrix given by

$$dG((u,v)) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} - \frac{-3u-v}{2\sqrt{16-3u^2-2uv-3v^2}} & -\frac{1}{2} - \frac{-u-3v}{2\sqrt{16-3u^2-2uv-3v^2}} \\ 0 & 1 \\ \frac{1}{2} - \frac{-3u-v}{2\sqrt{16-3u^2-2uv-3v^2}} & \frac{1}{2} - \frac{-u-3v}{2\sqrt{16-3u^2-2uv-3v^2}} \end{pmatrix}, \qquad dG((0,0)) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

dG is rank two because the first and third rows are independent, hence $G(U)$ is a parametrized 2-surface. The tangent space is $\mathbf{b} + dG((0,0))(\mathbf{R}^2)$.

(E7.) *Theorem. Suppose that* $f : \mathbf{R}^p \to \mathbf{R}$ *is* $\mathcal{C}^3$ *and some point* $\mathbf{a} \in \mathbf{R}^p$ *is critical* $df(\mathbf{a}) = 0$ *and* $d^2 f(\mathbf{a})$ *has both a positive and a negative eigenvalue. Then the critical point* $\mathbf{a}$ *is a saddle point: for every* $\delta > 0$ *there are points* $\mathbf{x}, \mathbf{y} \in B_\delta(\mathbf{a})$ *so that* $f(\mathbf{x}) < f(\mathbf{a}) < f(\mathbf{y})$.

By continuity of the third partial derivatives, a continuous function takes its maximum on a compact set, namely $M^2 = \sup \left\{ \sum_{i,j,k} \left(\frac{\partial^3 f(\mathbf{u})}{\partial x_i \partial x_j \partial x_k} \right)^2 : \|\mathbf{u} - \mathbf{a}\| \le 1 \right\}$. It follows that for any $\mathbf{h} \in \mathbf{R}^p$, and $\mathbf{c} \in B_1(\mathbf{a})$ that $|d^3 f(\mathbf{c})(\mathbf{h})^3| = \left| \sum_{i,j,k} \frac{\partial^3 f(\mathbf{c})}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k \right| \le M \|\mathbf{h}\|^3$ by applying the Schwarz inequality to each sum. Now by assumption there are $\mathbf{v}, \mathbf{w} \in \mathbf{R}^p$ unit eigenvectors such that $d^2 f(\mathbf{a})\mathbf{v} = \lambda_1 \mathbf{v}$ and $d^2 f(\mathbf{a})\mathbf{w} = \lambda_2 \mathbf{w}$ with $\lambda_1 < 0 < \lambda_2$. Assume that $0 < \delta < 1$ is so small that $M\delta < \min\{|\lambda_1|, |\lambda_2|\}$. Since second derivatives are differentiable, we can apply Taylor's formula to second order. We get for each $0 < t < \delta$ some $c \in (0,1)$ such that

$$\begin{aligned}
f(\mathbf{a}+t\mathbf{v}) &= f(\mathbf{a}) + df(\mathbf{a})(t\mathbf{v}) + \frac{1}{2}t^2\mathbf{v}\cdot d^2f(\mathbf{a})\mathbf{v} + \frac{1}{6}t^3 d^3f(\mathbf{a}+ct\mathbf{v})(\mathbf{v})^3 \\
&\le f(\mathbf{a}) + 0 + \frac{1}{2}\lambda_1 t^2\mathbf{v}\cdot\mathbf{v} + \frac{1}{6}t^3 M\|\mathbf{v}\|^3 \\
&\le f(\mathbf{a}) + \frac{1}{2}t^2\left(\lambda_1 + \frac{1}{3}tM\right) \le f(\mathbf{a}) + \frac{1}{2}t^2\left(\lambda_1 + \frac{1}{3}|\lambda_1|\right) < f(\mathbf{a}).
\end{aligned}$$

Similarly, for each $0 < t < \delta$, there is a $c \in (0,1)$ so that

$$\begin{aligned}
f(\mathbf{a}+t\mathbf{w}) &\ge f(\mathbf{a}) + \frac{1}{2}\lambda_2 t^2\mathbf{w}\cdot\mathbf{w} - \frac{1}{6}t^3M\|\mathbf{w}\|^3 \ge f(\mathbf{a}) + \frac{1}{2}t^2\left(\lambda_2 - \frac{1}{3}tM\right) \\
&\ge f(\mathbf{a}) + \frac{1}{2}t^2\left(\lambda_2 - \frac{1}{3}|\lambda_2|\right) > f(\mathbf{a}).
\end{aligned}$$

(E7*.) *Theorem. Assume that $f : \mathbf{R}^p \to \mathbf{R}$ is $\mathcal{C}^1$ function such that some point $\mathbf{a} \in \mathbf{R}^p$ is is a critical point: $df(\mathbf{a}) = 0$. Assume that the second partial derivatives exist in a neighborhood of a, are continuous at $\mathbf{a}$ and that the Hessian $d^2f(\mathbf{a})$ has both a positive and a negative eigenvalue at $\mathbf{a}$. Show that the critical point $\mathbf{a}$ is a saddle point: for every $\delta > 0$ there are points $\mathbf{x}, \mathbf{y} \in B_\delta(\mathbf{a})$ so that $f(\mathbf{x}) < f(\mathbf{a}) < f(\mathbf{y})$.*

Lemma. Suppose A and B are $p \times p$ real symmetric matrices such that A has both positive and negative eigenvalues $\lambda_1 < 0 < \lambda_2$ and such that the operator norm $\|A - B\| < \frac{1}{2}\min\{|\lambda_1|, \lambda_2\}$. Then the corresponding unit eigenvectors $\mathbf{v}_i$ of A satisfy $\mathbf{v}_1 \cdot B\mathbf{v}_1 < \frac{1}{2}\lambda_1$ and $\mathbf{v}_2 \cdot B\mathbf{v}_2 > \frac{1}{2}\lambda_1$.

Proof of Lemma. Let $\mathbf{v}_i$ be unit eigenvectors of A so that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for $i = 1, 2$. Then

$$\begin{aligned}
\mathbf{v}_1 \cdot B\mathbf{v}_1 &= \mathbf{v}_1 \cdot A\mathbf{v}_1 + \mathbf{v}_1 \cdot (B - A)\mathbf{v}_1 \\
&\le \lambda_1\|\mathbf{v}_1\|^2 + \|B - A\|\,\|\mathbf{v}_1\|^2 \\
&\le \lambda_1\|\mathbf{v}_1\|^2 + \frac{1}{2}|\lambda_1|\,\|\mathbf{v}_1\|^2 \le \frac{1}{2}\lambda_1\|\mathbf{v}_1\|^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbf{v}_2 \cdot B\mathbf{v}_2 &= \mathbf{v}_2 \cdot A\mathbf{v}_2 + \mathbf{v}_2 \cdot (B - A)\mathbf{v}_2 \\
&\ge \lambda_2\|\mathbf{v}_2\|^2 - \|B - A\|\,\|\mathbf{v}_2\|^2 \\
&\ge \lambda_2\|\mathbf{v}_2\|^2 - \frac{1}{2}|\lambda_2|\,\|\mathbf{v}_2\|^2 \ge \frac{1}{2}\lambda_2\|\mathbf{v}_2\|^2.
\end{aligned}$$

Proof of Theorem. By assumption there are $\mathbf{v}_i \in \mathbf{R}^p$ unit eigenvectors such that $d^2f(\mathbf{a})\mathbf{v}_i = \lambda_i\mathbf{v}_i$ with $\lambda_1 < 0 < \lambda_2$. Assume that $0 < \delta < 1$ is so small that the second partial derivatives of f exist on $B_\delta(\mathbf{a})$ and by continuity $\|d^2f(\mathbf{b}) - d^2f((\mathbf{a})\| < \frac{1}{2}\min\{|\lambda_1|, \lambda_2\}$ whenever $\mathbf{b} \in B_\delta(\mathbf{a})$. We apply the Lemma to $\mathbf{b} = \mathbf{a} + ct\mathbf{v}_i$. Since first derivatives are differentiable, we can apply Taylor's formula to first order. We get for each $0 < t < \delta$ some $c \in (0,1)$ such that at $\mathbf{b} = \mathbf{a} + ct\mathbf{v}_1$,

$$\begin{aligned}
f(\mathbf{a}+t\mathbf{v}_1) &= f(\mathbf{a}) + df(\mathbf{a})(t\mathbf{v}_1) + \frac{1}{2}t^2\mathbf{v}_1\cdot d^2f(\mathbf{a}+ct\mathbf{v}_1)\mathbf{v}_1 \\
&\le f(\mathbf{a}) + 0 + \frac{1}{4}\lambda_1 t^2\|\mathbf{v}_1\|^2 < f(\mathbf{a}).
\end{aligned}$$

Similarly, for each $0 < t < \delta$, there is a $c \in (0,1)$ so that at $\mathbf{b} = \mathbf{a} + ct\mathbf{v}_2$,

$$\begin{aligned}
f(\mathbf{a}+t\mathbf{v}_2) &= f(\mathbf{a}) + df(\mathbf{a})(t\mathbf{v}_2) + \frac{1}{2}t^2\mathbf{v}_2\cdot d^2f(\mathbf{a}+ct\mathbf{v}_2)\mathbf{v}_2 \\
&\ge f(\mathbf{a}) + 0 + \frac{1}{4}\lambda_2 t^2\|\mathbf{v}_2\|^2 > f(\mathbf{a}).
\end{aligned}$$

(E8*.) *(Slight generalization.) Theorem. Let* $F : \mathbf{R}^3 \to \mathbf{R}^3$ *be given by* $F(x,y,z) = (x^2 + y^2, xz, y^3 - z^3)$*. Then there is an open set* U *about* $P_0 = (3,1,2)$ *so that* F *is invertible on* U *and that* $F(U)$ *is an open set about* $Q_0 = F(P_0)$ *on which* F^{-1} *is* $\mathcal{C}^1$*. We find* $D[F^{-1}](Q)$ *where* $Q \in F(U)$*.*

The function $F(x,y,z)$ polynomial, therefore $\mathcal{C}^1$. We check that the determinant of the Jacobian matrix $\Delta_F(P_0) \neq 0$ and all of the conclusions follow from the Inverse Function Theorem. The fact that the linearization was invertible at the point enables you to conclude the existence of an inverse function. You were, however, given this in the problem.

Let $G \in \mathcal{C}^1(V, \mathbf{R}^3)$ be the inverse function of F. Thus in U we have the equation $F(G(x,y,z)) = (x,y,z)$. Apply the chain rule, and solve for DG at the point. Thus $D(F \circ G) = DF(P_0) \circ DG(Q_0) = I$ so $DG(Q_0) = (DF(P_0))^{-1}$. The matrix of $DF(P_0)$ is the Jacobian matrix

$$DF(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \\ z & 0 & x \\ 0 & 3y^2 & -3z^2 \end{pmatrix}; \quad DF(P_0) = \begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -12 \end{pmatrix}.$$

Thus $\Delta_F(P_0) = \det(DF_x(P_0)) = 6 \neq 0$. Finally, for Q near $(10, 6, -7) = Q_0 = F(P_0)$, so $DF^{-1}(Q) =$

$$[DF(F^{-1}(Q))]^{-1} = \frac{1}{6yz^3 - 6x^2y^2} \begin{pmatrix} -3xy^2 & 3z^3 & 3zy^2 \\ 6yz^2 & -6xz^2 & -6xy^2 \\ 2xy & -2x^2 & -2yz \end{pmatrix}; \quad DF^{-1}(Q_0) = \begin{pmatrix} \frac{3}{2} & -4 & -1 \\ -4 & 12 & 3 \\ -1 & 3 & \frac{2}{3} \end{pmatrix}$$

where $(x,y,z) = F^{-1}(Q)$. □

(E9*.) *(Slight generalization.) Theorem. Suppose* $G : \mathbf{R}^5 \to \mathbf{R}^3$ *is given by* $G(p,q,x,y,z) = (px + y^2, q^2z, py - qz + x)$. *Then there is an open set* U *around* $T_0 = (3,2)$ *and a* $\mathcal{C}^1$ *function* $H : U \to \mathbf{R}^3$ *so that* $H(3,2) = (1,5,4) = X_0$ *and for all* $(p,q) \in U$ *we have* $G(p,q,H(p,q)) = (28,16,8)$. *We find* $DH(3,2)$ *and* $DH(p,q)$.

The function G is polynomial so $\mathcal{C}^1$. We have to check that the linearization is soluble at $(3,2,1,5,4)$. Let $T_0 = (3,2)$ and $X_0 = (1,5,4)$. This follows if the $D_{\mathbf{x}}G$ part of the Jacobian matrix is invertible.

$$D_{\mathbf{x}}G = \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} & \frac{\partial G_1}{\partial z} \\ \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & \frac{\partial G_2}{\partial z} \\ \frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} & \frac{\partial G_3}{\partial z} \end{pmatrix} = \begin{pmatrix} p & 2y & 0 \\ 0 & 0 & q^2 \\ 1 & p & -q \end{pmatrix}; \quad D_{\mathbf{x}}G(T_0, X_0) = \begin{pmatrix} 3 & 10 & 0 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$$

which is invertible since its determinant is 4. The Implicit Function Theorem applies and yields the $\mathcal{C}^1$ function H as desired. (Again, you were given that there is $H \in \mathcal{C}^1(U)$ satisfying $F(p,q,H(p,q)) = (28,16,8)$ for all $(p,q) \in U$. Find the differential of H by differentiating the equation using the chain rule. Think of $\mathcal{H} : U \to U \times \mathbf{R}^3$ is given by $\mathcal{H}(p,q) = (p,q,H(p,q))$, and then differentiate $G \circ \mathcal{H} = \text{const}$ using the chain rule. Thus $D_{\mathbf{t}}\mathcal{H} = \binom{I}{D_{\mathbf{t}}H}$ so $0 = D_{\mathbf{t}}(G \circ \mathcal{H}) = D_{\mathbf{t}}G + D_{\mathbf{x}}G \circ D_{\mathbf{t}}H$. Here the total derivative matrix has columns associated to $\mathbf{t} = (p,q)$ and columns associated to $\mathbf{x} = (x,y,z)$ drivatives, $DG = (D_{\mathbf{t}}G, D_{\mathbf{x}}G)$. To find the total derivative of H we need the other part of the Jacobian

$$D_{\mathbf{t}}G = \begin{pmatrix} \frac{\partial G_1}{\partial p} & \frac{\partial G_1}{\partial q} \\ \frac{\partial G_2}{\partial p} & \frac{\partial G_2}{\partial q} \\ \frac{\partial G_3}{\partial p} & \frac{\partial G_3}{\partial q} \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 2qz \\ y & -z \end{pmatrix}; \qquad D_{\mathbf{t}}G(T_0, X_0) = \begin{pmatrix} 1 & 0 \\ 0 & 16 \\ 5 & -4 \end{pmatrix}; \qquad DH(T_0) = \begin{pmatrix} -3 & -16 \\ -\frac{25}{4} & -25 \\ -10 & 48 \end{pmatrix}$$

since total derivative of implicit function $DH(T) = -[D_{\mathbf{x}}G(T,H(T))]^{-1} D_{\mathbf{t}}G(T,H(T)) =$

$$\frac{1}{\Delta}\begin{pmatrix} pq^2 & q^2 & 0 \\ 2yq & -pq & 2y - p^2 \\ 2yq^2 & -pq^2 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 2qz \\ y & -z \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} xpq^2 & 2q^3z \\ 2xyq + 2y^2 - yp^2 & p^2z - 2pq^2z - 2yz \\ 2xyq^2 & -2pq^3z \end{pmatrix}$$

where $\Delta = q^2(p^2 - 2y)$ and $(x,y,z) = H(p,q)$. $\square$