Homework for Math 3220 §2, Spring 2013

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Our text is by Joseph L. Taylor, "Foundations of Analysis," American Mathematical Society, Pure and Applied Undergraduate Texts **18**, Providence, 2012. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on Apr. 24, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Please hand in problems A1 on Monday, Jan. 14.

A1. Euclidean Space. Exercises from Taylor's *Foundations of Analysis*

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167[4, 5, 6, 12]
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Please hand in problems B1 and B2 on Friday, Jan. 18.

B1. Sequences. Exercises from Taylor's *Foundations of Analysis*

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173[1, 8, 11]
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B2. Additional exercises.

1. Let (X, δ) be a metric space. Suppose that $x, y \in X$ and $\{x_n\}$ and $\{y_n\}$ are convergent sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Show that $\delta(x_n, y_n) \to \delta(x, y)$ as $n \to \infty$.
2. Let v be a vector and $\{u_n\}$ be a sequence in $\mathbb{R}^d$. Suppose that $\{u_n\}$ is *weakly convergent* to v, that is, for every $w \in \mathbb{R}^d$ we have $w \cdot u_n \to w \cdot v$ as $n \to \infty$. Show that then it is convergent: $u_n \to v$ as $n \to \infty$.

Please hand in problems C1 and C2 on Friday, Jan. 25.

C1. Open and Closed Sets. Exercises from Taylor's *Foundations of Analysis* §7.3.

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178[1, 2, 8, 10]
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C2. Additional exercises.

1. Let $a, b, c \in \mathbb{R}$ so that $a^2 + b^2 = 1$. Let
$$\mathcal{L} = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$
consist of points of a line in the plane. Show that $\mathcal{L}$ is a closed set.
2. Let
$$S = \left\{ \left(\frac{1}{n}, \frac{1}{n} \right) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}.$$
Determine whether S is closed, open or neither. Prove your assertion.
3. Let $E \subset \mathbb{R}^d$. Show that the closure is given by
$$\overline{E} = \left\{ x \in \mathbb{R}^d : (\forall r > 0)\,(B_r(x) \cap E \neq \emptyset) \right\}.$$

Please hand in problems D1 on Friday, Feb. 1.

D1. Compact Sets. Exercises from Taylor's *Foundations of Analysis* §7.4.

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182[1, 4, 10, 11]
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Please hand in problems E1 and E2 on Friday, Feb. 8.

E1. Connected Sets. Exercises from Taylor's *Foundations of Analysis* §7.5, §8.1.

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188[2, 6, 11],
195[3, 5]
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E2. Additional exercises.

1. Prove that the "topologist's sine curve" S in the Euclidean plane is connected.
$$S = \left\{ \left(x, \sin \frac{1}{x} \right) : 0 < x < 1 \right\} \cup \{(0, y) : -1 \leq y \leq 1\}.$$

Please hand in problems F1 and F2 on Friday, Feb. 15.

F1. Continuous Functions. Exercises from Taylor's *Foundations of Analysis* §8.1, §8.2.

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195[11],
201[6, 7, 10, 11]
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F2. Additional exercises.

1. Suppose I and J are open intervals in $\mathbb{R}$, $a \in I$ and $b \in J$. Suppose that $f : I \times J - \{(a,b)\} \to \mathbb{R}$ is a function such that for all $x \in I - \{a\}$ the limit exists

   $$g(x) = \lim_{y \to b} f(x,y)$$

   and that for all $y \in J - \{b\}$ the limit exists:

   $$h(y) = \lim_{x \to a} f(x,y).$$

   Show that even though the "iterated limits" may exist

   $$L = \lim_{x \to a} g(x), \qquad M = \lim_{y \to b} h(y),$$

   it may be the case that $L \neq M$. Show, then, that the two-dimensional limit

   $$\lim_{(x,y)\to(a,b)} f(x,y)$$

   fails to exist. Suppose in addition to the existence of the iterated limits one knows that the two dimensional limit exists: $f(x,y) \to N$ as $(x,y) \to (a,b)$. Show that then $L = M = N$.

2. Suppose that $E \subset \mathbb{R}^p$ is a bounded set. Show that there is a closed ball containing E which has minimal radius.

Please hand in problems G1 on Friday, Feb. 22.

G1. Uniform Convergence. Exercises from Taylor's *Foundations of Analysis* §8.3 – §8.5. Read the review sections §8.3 and §8.4 about linear algebra. Do any problem whose solution isn't immediately clear.

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206[1, 7, 8, 10],
214[14],
221[10].
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Please hand in problems H1 on Friday, Mar. 1.

H1. Differentiation. Exercises from Taylor's *Foundations of Analysis* §9.1.

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228[1, 8, 9, 10],
235[3].
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Please hand in problems I1 and I2 on Friday, Mar. 8.

I1. Differentiable Functions. Exercises from Taylor's *Foundations of Analysis* §9.2-4.

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235[4, 5],
241[1, 5, 10],
250[1, 4, 11].
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I2. (See 9.3[9]) Suppose that (x, y, z) are the Cartesian coordinates of a point in $\mathbb{R}^3$ and the spherical coordinates of the same point is given by

$$\begin{aligned} x &= r\cos\vartheta\,\sin\varphi, \\ y &= r\sin\vartheta\,\sin\varphi, \\ z &= r\cos\varphi. \end{aligned}$$

Let $u = f(x, y, z)$ be a $\mathcal{C}^2$ function on $\mathbb{R}^3$. Find a formula for the partial derivatives of u with respect to x, y, z in terms of partial derivatives with respect to r, ϑ, φ. Find a formula for the Laplacian of u in terms of partial derivatives with respect to r, ϑ, φ, where the Laplacian is given by

$$\Delta\, u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Please hand in problems J1 – J4 on Friday, Mar. 22.

J1. Brook Taylor's Formula. Exercises from Joseph Taylor's *Foundations of Analysis* §9.5.

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259[3, 6, 9].
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J2. Find the critical point (s_0, t_0) in the set $\{(s, t) \in \mathbb{R}^2 : s > 0\}$ for the function with any real A and $B > 0$,

$$f(s, t) = \log(s) + \frac{(t - A)^2 + B^2}{s}.$$

Find the second order Taylor's expansion for f about the point (s_0, t_0). Prove that f has a local minimum at (s_0, t_0).

J3. Find all extrema of the function $f(x) = x_1^2 + \cdots + x_n^2$ subject to the constraint $|x_1|^p + \cdots + |x_n|^p = 1$. If $1 \le p \le 2$ show for any x and n that

$$n^{\frac{p-2}{2p}} \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}} \le \sqrt{x_1^2 + \cdots + x_n^2} \le \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}.$$

J4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\begin{aligned} x &= u^2 - v^2, \\ y &= 2uv; \end{aligned}$$

Find an open set $U \subset \mathbb{R}^2$ such that $(3, 4) \in U$ and $V = F(U)$ is an open set for which there is a $\mathcal{C}^1$ function $G : V \to U$ such that

$$G \circ F(u, v) = (u, v) \quad \text{for all } (u, v) \in U \text{ and } \quad F \circ G(x, y) = (x, y) \quad \text{for all } (x, y) \in V.$$

Find the differential $dG(F(3, 4))$.

(G is a local inverse. Solve for G and check its properties. Do not use the Inverse Function Theorem, which guarantees the existence of local inverse near $(3, 4)$ assuming F is continuously differentiable near $(3, 4)$ and $dF(3, 4)$ is invertible.)

Please hand in problem K1 on Friday, Mar. 29.

K1. Inverse Function Theorem. Exercises from Taylor's *Foundations of Analysis* §9.6.

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265[2, 8]. (Postponed from last week.)
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Please hand in problem L1 – L3 on Friday, Apr. 5.

L1. Implicit Function Theorem. Exercises from Taylor's *Foundations of Analysis* §9.7.

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272[1, 5, 8]. (Postponed from last week),
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L2. In section §9.7 the Implicit Function Theorem was deduced from the Inverse Function Theorem. Show that the Inverse Function Theorem can be deduced from the Implicit Function Theorem.

L3. Suppose that $V \subset \mathbb{R}^p$ is an open set, that $\mathbf{a} \in V$ and that $f \in \mathcal{C}^1(V)$. If $f(\mathbf{a}) = 0$, $\frac{\partial f}{\partial x_j}(\mathbf{a}) \neq 0$ and $\mathbf{u}^{(j)} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$ for $j = 1, \ldots, p$, prove that there exist open sets W_j containing $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_p)$, functions $g_j(\mathbf{u}^{(j)})$ which are $\mathcal{C}^1$ on W_j such that

$$f(x_1, x_2, \ldots, x_{j-1}, g_j(\mathbf{u}^{(j)}), x_{j+1}, \ldots, x_p) = 0$$

for all $\mathbf{u}^{(j)} \in W_j$. Moreover, for some $r > 0$, if $\mathbf{x} \in B_r(\mathbf{a})$ such that $f(\mathbf{x}) = 0$ we have

$$\frac{\partial g_1}{\partial x_p}(\mathbf{u}^{(1)})\,\frac{\partial g_2}{\partial x_1}(\mathbf{u}^{(2)})\,\frac{\partial g_3}{\partial x_2}(\mathbf{u}^{(3)})\cdots\frac{\partial g_p}{\partial x_{p-1}}(\mathbf{u}^{(p)}) = (-1)^p.$$

Please hand in problem M1 – M2 on Friday, Apr. 12.

M1. Riemann Integration. Exercises from Taylor's *Foundations of Analysis* §10.1–10.2.

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282[5, 6, 9],
287[2, 4, 5, 12].
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M2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a continuously differentiable curve. Show that the image of a compact interval $\gamma([a, b])$ has Jordan content zero, called "volume zero" in the text.

(Note that this would be false under the hypothesis of continuity only. This is because there exist "space filling curves." See, e.g.,

`http://www.math.ohio-state.edu/~fiedorow/math655/Peano.html.)`

Please hand in problem N1 on Friday, Apr. 19. This is the last homework assignment for the semester. All outstanding homework is due Apr. 19.

N1. Fubini's Theorem. Exercises from Taylor's *Foundations of Analysis* §10.3–10.5.

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293[5, 11],
302[2, 7, 9, 10],
314[2, 7, 8, 11, 12].
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