Math 3220 § 1. **Second Midterm Exam** Name: Solutions
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(1.) *Consider the subset of* $\mathbf{R}^2$ *given by*

$$E = \left\{(x,y) : x < 0\right\} \cup \left\{\left(\frac{1}{2},0\right), \left(\frac{2}{3},0\right), \left(\frac{3}{4},0\right), \dots\right\}.$$

(a.) Find E°.

$E^\circ = \{(x,y) : x < 0\}$. For the other points $(\frac{1}{n},0) \in E$, for no $r > 0$ is $B_r((\frac{1}{n},0)) \subseteq E$.

(b.) *Find* $\overline{E}$.

$\overline{E} = \{(x,y) : x \le 0\} \cup \{(\frac{1}{2},0),(\frac{2}{3},0),(\frac{3}{4},0),\dots\} \cup \{(1,0)\}$. These are points $\mathbf{z} \in \mathbf{R}^2$ such that for all $r > 0$, $B_r(\mathbf{z}) \cap E \ne \emptyset$.

(c.) *Find* ∂E.

$\partial E = \{(0,y) : y \in \mathbf{R}\} \cup \{(\frac{1}{2},0),(\frac{2}{3},0),(\frac{3}{4},0),\dots\} \cup \{(1,0)\}$. These are points $\mathbf{z} \in \mathbf{R}^2$ such that for all $r > 0$, $B_r(\mathbf{z}) \cap E \ne \emptyset$ and $B_r(\mathbf{z}) \cap E^c \ne \emptyset$.

(d.) *Determine whether* E *is connected.*

E is not connected. Consider the open sets $U = \{(x,y) : x < 0\}$ and $V = \{(x,y) : \frac{1}{4} < x\}$. Then $U \cap V = \emptyset$, $U \cap E \ne \emptyset$, $V \cap E \ne \emptyset$ and $E \subseteq U \cup V$, so U and V disconnect E.

(2.) *Define:* $\mathcal{H} \subseteq \mathbf{R}^n$ *is an open set. Let* $\mathbf{v} \in \mathbf{R}^n$ *be a vector with unit length* $\|\mathbf{v}\| = 1$ *and* $c \in \mathbf{R}$. *Using the definition, show that* $\mathcal{H}$ *is open, where* $\mathcal{H} = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x}\cdot\mathbf{v} > c\}$

Definition: $\mathcal{H} \subseteq \mathbf{R}^n$ is *open* if for all $\mathbf{x} \in \mathcal{H}$, there is $r > 0$ so that $B_r(\mathbf{x}) \subseteq \mathcal{H}$.

The set $\mathcal{H}$ is an open halfplane. For any point $\mathbf{x} \in \mathcal{H}$, the largest ball about $\mathbf{x}$ which is still in $\mathcal{H}$ has a radius equal to the distance from $\mathbf{x}$ to $\partial\mathcal{H}$, namely, let $\varepsilon = \mathbf{x}\cdot\mathbf{v} - c$ which is positive because being in $\mathcal{H}$ means $\mathbf{x}\cdot\mathbf{v} > c$. I claim that $B_\varepsilon(x) \subseteq \mathcal{H}$. To see this, we choose $\mathbf{z} \in B_\varepsilon(\mathbf{x})$ to show that $\mathbf{z} \in \mathcal{H}$. This follows from and the Cauchy-Schwarz Inequality:

$$\mathbf{z}\cdot\mathbf{v} = \mathbf{x}\cdot\mathbf{v} + (\mathbf{z}-\mathbf{x})\cdot\mathbf{v} \ge \mathbf{x}\cdot\mathbf{v} - \|\mathbf{z}-\mathbf{x}\|\,\|\mathbf{v}\| > \mathbf{x}\cdot\mathbf{v} - \varepsilon\cdot 1 = \mathbf{x}\cdot\mathbf{v} - (\mathbf{x}\cdot\mathbf{v} - c) = c.$$

(3.) *Suppose that* $\{\mathbf{b}_k\}_{k\in\mathbf{N}}$ *is a bounded sequence and* $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ *converges to zero in* $\mathbf{R}^3$. *Show that the sequence of cross products converges and* $\lim_{k\to\infty}(\mathbf{b}_k \times \mathbf{x}_k) = \mathbf{0}$.

The argument follows the $\mathbf{R}^1$ proof for products, except that we use the inequality $\|\mathbf{x}\times\mathbf{b}\| = |\sin\theta|\,\|\mathbf{x}\|\,\|\mathbf{b}\| \le \|\mathbf{x}\|\,\|\mathbf{b}\|$, where $\theta = \angle(\mathbf{b},\mathbf{x})$ is the angle between the vectors.

We are given that the sequence $\{\mathbf{b}_k\}$ is bounded. Thus, there is an $M < \infty$ so that $\|\mathbf{b}_k\| \le M$ for all k. We are also given that $\mathbf{x}_k \to \mathbf{0}$ as $k \to \infty$. Thus, for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \ge N$ we have $\|\mathbf{x}_k - \mathbf{0}\| < \dfrac{\varepsilon}{1+M}$. For this same N we have for all $k \ge N$, using the inequality,

$$\|\mathbf{b}_k \times \mathbf{x}_k - \mathbf{0}\| = \|\mathbf{b}_k \times \mathbf{x}_k\| \le \|\mathbf{b}_k\|\,\|\mathbf{x}_k\| \le M\cdot\frac{\varepsilon}{1+M} < \varepsilon.$$

Thus we have shown that $\mathbf{b}_k \times \mathbf{x}_k \to \mathbf{0}$ as $k \to \infty$.

(4.) *Determine whether the statement is true or false.*

(a.) **Statement.** *Let* $A, B \subseteq \mathbf{R}^n$. *If* A *is closed and* B *is open then* $A\backslash B$ *is closed.*

TRUE! Since B is open, the complement B^c is closed. However $A\backslash B = A \cap B^c$ is closed because it is the intersection of closed sets.

(b.) **Statement.** *If a set* $E \subseteq \mathbf{R}^n$ *is not closed then it equals its own interior* $E = E^\circ$.

FALSE! The set $E = (0,1] \subseteq \mathbf{R}$ is not closed because it does not contain one of its limit points: $\frac{1}{n} \in E$ for $n \in \mathbf{N}$ and $\frac{1}{n} \to 0$, a limit point of E as $n \to \infty$, but $0 \notin E$. However, the interior is $E^\circ = (0,1) \ne E$.

(c.) **Statement.** *Suppose* $E \subseteq \mathbf{R}$ *has connected closure and connected boundary. Then* E *is connected.*

FALSE! Consider the set $E = \mathbf{Q} \subseteq \mathbf{R}$, the set of rational numbers. The closure $\overline{E} = \mathbf{R}$ and the boundary $\partial E = \mathbf{R}$ so both are connected, as they are intervals of $\mathbf{R}$, but $U = \{x : x < \sqrt{2}\}$

and $V = \{x : x > \sqrt{2}\}$ disconnect E: U and V are open sets in $\mathbf{R}$ such that $U \cap V = \emptyset$, $U \cap E \neq \emptyset$, $V \cap E \neq \emptyset$ and $E \subseteq U \cup V$.

(5.) *Suppose that a subset* $A \subseteq E \subseteq \mathbf{R}^n$. *Show that* A *is relatively open in* E *if and only if the condition* (O.N.) *holds:*

$$(\forall \mathbf{x} \in A)(\exists G \subseteq \mathbf{R}^n : \ G \textit{ is an open set })(\mathbf{x} \in G \cap E \subseteq A). \qquad \text{O.N.}$$

A subset A is *relatively open* in E if there is an open set $\mathcal{O} \subseteq \mathbf{R}^n$ such that $A = \mathcal{O} \cap E$. Being relatively open trivially implies the condition (O.N.) If A is relatively open, let $\mathcal{O} \subseteq \mathbf{R}^n$ be the open set such that $A = \mathcal{O} \cap E$. Then for every $\mathbf{x} \in A$, we may take $G = \mathcal{O}$, because then $\mathbf{x} \in A = G \cap E \subseteq E$ satisfies the condition (O.N.)

Now to show that the condition (O.N.) implies that A is relatively open, we have to construct an open set $\mathcal{G} \subseteq \mathbf{R}^n$ so that $A = \mathcal{G} \cap E$. For each $\mathbf{x} \in A$ let $G_{\mathbf{x}} \subseteq \mathbf{R}^n$ be the open set in the definition of (O.N.), such that $\mathbf{x} \in G_{\mathbf{x}} \cap E \subseteq A$. Let $\mathcal{G} = \bigcup_{\mathbf{x} \in A} G_{\mathbf{x}}$. As this is the union of open sets, $\mathcal{G}$ is also open. With this set, I claim that A is relatively open, namely $A = \mathcal{G} \cap E$. To see "$\supseteq$," suppose that $\mathbf{y} \in \mathcal{G} \cap E$. This means that there is an $\mathbf{x} \in A$ so that $\mathbf{y} \in G_{\mathbf{x}}$. But by the construction of $G_{\mathbf{x}}$ and since $\mathbf{y} \in E$ we have $\mathbf{y} \in G_{\mathbf{x}} \cap E \subseteq A$. To see "$\subseteq$," choose $\mathbf{z} \in A$. Again, by the construction of $G_{\mathbf{x}}$, $\mathbf{z} \in G_{\mathbf{z}} \cap E \subseteq \mathcal{G} \cap E$, as $\mathcal{G}$ is the union of such sets.