5010 solutions, Assignment 2. Chapter 2: 1, 3, 5–9.

1. Given $P(W) = 0.3$, $P(H \mid W) = 0.4$, and $P(H \mid W^c) = 0.7$.
(a) $P(W \cap H) = P(W)P(H \mid W) = (0.3)(0.4) = 0.12$.
(b) $P(H) = P(W)P(H \mid W) + P(W^c)P(H \mid W^c) = (0.3)(0.4) + (0.7)(0.7) = 0.61$.
(c) $(0.61)(0.39) + (0.39)(0.61) = 0.4758$.
(d) $P(W^c \mid H^c) = P(W^c)P(H^c \mid W^c)/P(H^c) = (0.7)(1 - 0.7)/(1 - 0.61) = 21/39 = 7/13$.

3. Given $P(H \mid C_1) = 1/3$, $P(H \mid C_2) = 2/3$, and $P(H \mid C_3) = 1$.
(a) $P(H) = P(C_1)P(H \mid C_1) + P(C_2)P(H \mid C_2) + P(C_3)P(H \mid C_3) = (1/3 + 2/3 + 1)/3 = 2/3$, so $P(C_1 \mid H) = P(C_1)P(H \mid C_1)/P(H) = (1/3)(1/3)/(2/3) = 1/6$, $P(C_2 \mid H) = P(C_2)P(H \mid C_2)/P(H) = (1/3)(2/3)/(2/3) = 1/3$, and $P(C_3 \mid H) = P(C_3)P(H \mid C_3)/P(H) = (1/3)(1)/(2/3) = 1/2$.
(b) $P(H_2 \mid H_1) = P(H_1 \cap H_2)/P(H_1)$ and $P(H_1) = 2/3$ from (a). $P(H_1 \cap H_2) = P(C_1)P(H_1 \cap H_2 \mid C_1) + P(C_2)P(H_1 \cap H_2 \mid C_2) + P(C_3)P(H_1 \cap H_2 \mid C_3) = ((1/3)^2 + (2/3)^2 + 1^2)/3 = 14/27$, so $P(H_2 \mid H_1) = (14/27)/(2/3) = 7/9$.
(c) $P(HH) = P(C_1)P(HH \mid C_1) + P(C_2)P(HH \mid C_2) + P(C_3)P(HH \mid C_3) = ((1/3)^2 + (2/3)^2 + 1^2)/3 = 14/27$, so $P(C_1 \mid HH) = P(C_1)P(HH \mid C_1)/P(HH) = (1/3)(1/3)^2/(14/27) = 1/14$, $P(C_2 \mid HH) = P(C_2)P(HH \mid C_2)/P(HH) = (1/3)(2/3)^2/(14/27) = 2/7$, and $P(C_3 \mid HH) = P(C_3)P(HH \mid C_3)/P(HH) = (1/3)(1)^2/(14/27) = 9/14$.
(d) $P(H_3 \mid HH) = P(HH \cap H_3)/P(HH)$ and $P(HH) = 14/27$ from (c). $P(HH \cap H_3) = P(C_1)P(HH \cap H_3 \mid C_1) + P(C_2)P(HH \cap H_3 \mid C_2) + P(C_3)P(HH \cap H_3 \mid C_3) = ((1/3)^3 + (2/3)^3 + 1^3)/3 = 4/9$, so $P(H_3 \mid HH) = (4/9)/(14/27) = 6/7$.

5. (a) $P(F_1 \cap P_2) = P(F_1)P(P_2 \mid F_1) = p_1(1 - p_2) = (0.6)(1 - 0.4) = 0.36$.
(b) $P(F_1 \cap F_2 \cap P_3) = P(F_1)P(F_2 \mid F_1)P(P_3 \mid F_1 \cap F_2) = p_1 p_2 (1 - p_3) = (0.6)(0.4)(1 - 0.75) = 0.06$.
(c) $P(P_2 \cup (F_2 \cap P_3) \mid F_1) = P(P_2 \mid F_1) + P(F_2 \cap P_3 \mid F_1) = 1 - p_2 + p_2(1 - p_3) = 1 - 0.4 + (0.4)(1 - 0.75) = 0.7$.
(d) $P(F_1 \cap P_2 \mid P_1 \cup (F_1 \cap P_2) \cup (F_1 \cap F_2 \cap P_3)) = p_1(1 - p_2)/[1 - p_1 + p_1(1 - p_2) + p_1 p_2 (1 - p_3)] = (0.6)(1 - 0.4)/[1 - 0.6 + (0.6)(1 - 0.4) + (0.6)(0.4)(1 - 0.75)] = 0.36/0.82 = 18/41$.

6. The sum of 3 can occur in only one way: 2 on the first die, 1 on the second. (If a 1 appears on the first die, the total is 1. If a 3 appears on the first die, the total is at least 5.) (a) 0. (b) 1.

7. (a) He chooses a face at random. Of the 10 faces, there are 6 heads and 4 tails. So $6/10 = 3/5$ is the answer.
(b) He sees one of the faces that is a head. It is one of 6 such. 4 of those heads are on double-headed coins, so the answer is $4/6 = 2/3$.
(c) He tosses and sees a head. What is the probability that the next toss produces a head? $P(H_2 \mid H_1) = P(H_1 \cap H_2)/P(H_1) = [(2/5)(1) + (1/5)(0) + (2/5)(1/4)]/[(2/5)(1) + (1/5)(0) + (2/5)(1/2)] = (1/2)/(3/5) = 5/6$.

(d) He tosses again and sees another head. $P(HH \mid H_1 \cap H_2) = P(HH)P(H_1 \cap H_2 \mid HH)/P(H_1 \cap H_2) = (2/5)(1)/[(2/5)(1) + (1/5)(0) + (2/5)(1/4)] = 4/5$.

8. (a) $2/4 = 1/2$.
(b) $2/4 = 1/2$; $1/4$.
(c) Both must be blue and the same number. There are $4 \cdot 3$ ways to choose the two dice (taking order into account) and only 2 ways have both dice blue. There are 36 ways to choose the two numbers and only 6 ways have both numbers the same. So the answer is $(2)(6)/[(4)(3)(36)] = 1/36$.
(d) We assume "did not show the same number and color" means "showed a different number or different color." $P(S_2 \mid S_1^c) = P(S_1^c \cap S_2)/P(S_1^c)$ and $P(S_1^c) = 35/36$ by (c). Now for the event $S_1^c \cap S_2$ to occur, the 3rd and 4th dice must be blue and the same number, and the 1st and 2nd will automatically be different colors. The probability is $1/36$, by analogy with (c). So the conditional probability is $(1/36)/(35/36) = 1/35$.

9. (a) $P(G_1) = P(A)P(G_1 \mid A) + P(B)P(G_1 \mid B) = (1/2)(9/12) + (1/2)(3/12) = 1/2$.
(b) $P(G_1 \mid A) = P(A)P(G_1 \mid A)/P(G_1) = (1/2)(9/12)/(1/2) = 3/4$.
(c) $P(G_3 \mid G_1 \cap G_2) = P(G_1 \cap G_2 \cap G_3)/P(G_1 \cap G_2) = (1/2)((9/12)^3 + (3/12)^3)/[(1/2)((9/12)^2 + (3/12)^2)] = (28/64)/(10/16) = 7/10$.