5010 solutions, Assignment 4. Chapter 3: 1, 2, 4, 6, 9–11.

1.
$$\frac{\binom{4}{2}+\binom{6}{2}+\binom{8}{2}}{\binom{4+6+8}{2}} = \frac{6+15+28}{153} = \frac{49}{153}.$$

2. (a)
$$\frac{a!\,b!\,c!\,3!}{(a+b+c)!}.$$

(b)
$$\frac{(a+b+c)!}{a!\,b!\,c!}\Big/(a+b+c)! = \frac{1}{a!\,b!\,c!}.$$

(c)
$$\frac{3!}{(a+b+c)!}.$$

4. (a)
$$\frac{\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}^3}{\binom{52}{5}}.$$

(b)
$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}^2\binom{4}{1}}{\binom{52}{5}}.$$

(c)
$$\frac{\binom{10}{1}\left(\binom{4}{1}^5-\binom{4}{1}\right)}{\binom{52}{5}}.$$

(d)
$$\frac{\left(\binom{13}{5}-\binom{10}{1}\right)\binom{4}{1}}{\binom{52}{5}}.$$

(e)
$$\frac{(13)_2\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}.$$

In parts (c) and (d), we have ruled out straight flushes, which the book inludes.

6. The key observation is that numbers do not begin with a 0, with the exception of 0 itself. So we treat several cases, according to the number of digits.

One-digit numbers: 0–9. There are 10.

Two digit numbers: First digit is 1–9, second is 0–9 but different. There are $9 \cdot 9 = 81$.

Three digit numbers: $9 \cdot 9 \cdot 8 = 648$.

Four-digit numbers: $9 \cdot 9 \cdot 8 \cdot 7 = 4{,}536$.

Five-digit numbers: $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27{,}216$.
Six-digit numbers: $10^5 = 100{,}000$, but digits are repeated, so none.
Total is 32,491. Answer is 32,491/100,001.

9. As usual, think of the dice as distinguishable. There are $6^5 = 7{,}776$ possible outcomes. Note the ambiguous wording: "includes" and "consists of" have different meanings.

(a) "Includes four aces" means four or five aces. Exactly four aces has probability $\binom{5}{1}(1/6)^4(5/6)^1 = 25/6^5$, and exactly five aces has probability $\binom{5}{0}(1/6)^5(5/6)^0 = 1/6^5$. Answer is $26/6^5$.

(b) Answer is 6 times the answer to (a).

(c) Let A be the event "at least one ace," K the event "at least one king, and Q the event "at least one queen." We are interested in $P(A \cap K \cap Q) = 1 - P(A^c \cup K^c \cup Q^c)$. This is

$$
\begin{aligned}
&P((A^c \cup K^c) \cup Q^c) \\
&= P(A^c \cup K^c) + P(Q^c) - P((A^c \cup K^c) \cap Q^c) \\
&= P(A^c) + P(K^c) - P(A^c \cap K^c) + P(Q^c) - P((A^c \cap Q^c) \cup (K^c \cap Q^c)) \\
&= P(A^c) + P(K^c) - P(A^c \cap K^c) + P(Q^c) - [P(A^c \cap Q^c) + P(K^c \cap Q^c) - P(A^c \cap K^c \cap Q^c)] \\
&= 3P(A^c) - 3P(A^c \cap K^c) + P(A^c \cap K^c \cap Q^c) \\
&= 3(5/6)^5 - 3(4/6)^5 + (3/6)^5
\end{aligned}
$$

so the answer is $1 - [3(5/6)^5 - 3(4/6)^5 + (3/6)^5]$.

10. (a) $(8+8+2)/\binom{64}{8}$.

(b) $8^2 \cdot 7^2 \cdot 6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2/(64)_8 = 8!/\binom{64}{8}$.

11. (a) $= 1 - P(\text{none is black}) = \binom{n}{0}\binom{3n}{r}/\binom{4n}{r}$.

(b) $\binom{n}{2}\binom{3n}{r-2}/\binom{4n}{r}$.

(c) This one will be easier with the inclusion-exclusion law, which we will cover on Wednesday.