5010 solutions, Assignment 7. Chapter 4: 30, 33, 35, 40, 42, 45. Chapter 5: 2, 14, 15.

30. (a) Condition on whether day $n-1$ was wet or fine.

$$u_n = u_{n-1}p + (1-u_{n-1})p' = (p-p')u_{n-1} + p', \qquad n \geq 2.$$

We can iterate to get

$$\begin{aligned} u_n &= (p-p')u_{n-1} + p' = (p-p')[(p-p')u_{n-2} + p'] + p' = (p-p')^2 u_{n-2} + (p-p')p' + p' \\ &= (p-p')^2[(p-p')u_{n-3} + p'] + (p-p')p' + p' = (p-p')^3 u_{n-3} + (p-p')^2 p' + (p-p')p' + p' \\ &\vdots \\ &= (p-p')^n u_0 + \sum_{k=0}^{n-1}(p-p')^k p' \to \sum_{k=0}^{\infty}(p-p')^k p' = \frac{p'}{1-(p-p')}. \end{aligned}$$

(b) This is a geometric random variable with parameter $1-p$ (i.e., it is the number of days until the first success, where success means a wet day), hence the mean is $1/(1-p)$.

(c) The situation is like this: (f, today)ff. . . fwf ff. . . fwf ff. . . fwf . . . ff. . . fww. Each ff. . . fwf requires $1/(1-p)+1$ days on average by part (b). The number of such patterns has a geometric$(1-p')$ distribution, which has mean $1/(1-p')$. The product is

$$\left(\frac{1}{1-p} + 1\right)\frac{1}{1-p'} = \frac{2-p}{(1-p)(1-p')}.$$

33. (a) Let X_1 be her profit from game 1 and X_2 be her profit from game 2. The $P(X_1 = a) = 0.4$ and $P(X_1 = -a) = 0.6$. Similarly, $P(X_1 = b) = 0.4$ and $P(X_1 = -b) = 0.6$. Then $E[X_1 + X_2] = E[X_1] + E[X_2] = a(0.4-0.6) + b(0.4-0.6) = (a+b)(-0.2) = -0.2$.

(b) Now $P(X_1 = b) = ap$ and $P(X_1 = -b) = 1-ap$, so $E[X_1 + X_2] = E[X_1] + E[X_2] = a(0.4-0.6) + b(ap-(1-ap)) = -0.2a + 2bap - b$. Since $b = 1-a$, this equals $f(a) = -0.2a + (2ap-1)(1-a)$. This function is maximized at a critical point, i.e., $0 = f'(a) = -0.2 - (2ap-1) + 2p(1-a)$. Solving, we get $a = (-0.2+1+2p)/(2p+2p) = 0.5 + 0.2/p$.

35. (a) $P(X = r) = P($first $k-1+r$ oysters contain $k-1$ pearls, $(k+r)$th oyster contains kth pearl$) = \binom{k-1+r}{k-1}p^k(1-p)^r$. Sum over $r \geq 0$ and use the negative binomial theorem (page 22) to get

$$\sum_{r=0}^{\infty}\binom{k-1+r}{k-1}p^k(1-p)^r = p^k p^{-k} = 1.$$

(b) Mean is

$$\sum_{r=0}^{\infty}\binom{k-1+r}{k-1} r p^k(1-p)^r = \sum_{r=1}^{\infty}\frac{(k-1+r)_r}{r!} r p^k(1-p)^r$$

$$= \sum_{r=1}^{\infty} \frac{(k-1+r)_{r-1}}{(r-1)!} kp^k(1-p)^r = \sum_{r=1}^{\infty} \binom{k+r-1}{r-1} kp^k(1-p)^r$$
$$= \sum_{r=0}^{\infty} \binom{k+r}{r} kp^k(1-p)^{r+1} = kp^k(1-p)p^{-(k+1)} = k(p^{-1}-1).$$

The variance can be found in the same way, but, as we will soon see, there are much simpler methods to find the mean and variance.

(c) $\binom{k-1+r}{k-1}p^k(1-p)^r = \binom{k-1+r}{r}(1-\lambda/k)^k(\lambda/k)^r \to e^{-\lambda}\lambda^r/r!$, the Poisson distribution.

40. $P(X \geq a+1) = P(e^{tX} \geq e^{t(a+1)}) \leq E[e^{tX}]e^{-t(a+1)}$. Now the mgf of the geometric distribution is $E[e^{tX}] = \sum_1^{\infty} e^{tn}q^{n-1}p = pe^t/(1-qe^t)$, so we get $P(X \geq a+1) \leq pe^{-ta}/(1-qe^t)$. Since this is valid for every t, we want to choose t to minimize this upper bound. By taking the derivative, we find that the minimum is at $qe^t = a/(a+1)$, hence $pe^{-ta} = p(q(a+1)/a)^a$, and we find that $P(X \geq a+1) \leq (a+1)p(q(a+1)/a)^a$. Now the exact value of $P(X \geq a+1)$ is q^a, so our bound exceeds it by a factor of $(a+1)p(1+1/a)^a$.

42. (a) $P(|X-\mu| \leq h\sigma) = 1 - P(|X-\mu| > h\sigma) = 1 - P((X-\mu)^2 > h^2\sigma^2) \geq 1 - \sigma^2/(h^2\sigma^2) = 1 - 1/h^2$.

(b) Denote m by S_n, which looks more like a random variable. In fact S_n is binomial$(n, 1/2)$, so the mean and variance of S_n/n are $1/2$ and $1/(4n)$. If $n \geq 100$, then $P(0.4 \leq S_n/n \leq 0.6) \geq P(|S_n/n - 0.5| \leq h/(2\sqrt{n})) \geq 1 - 1/h^2$, provided $0.1 \geq h/(2\sqrt{n})$, and this holds for $h = 2$ since $n \geq 100$. The result follows.

(c) This is $P(S_n \in \{49, 50, 51\}) = \sum_{k=49}^{51} \binom{100}{k} 2^{-100}$, which can be estimated by Stirling's formula.

45. Let the probabilities of the three faces be $p_1 = 1/2$, $p_2 = 1/3$, and $p_3 = 1/6$. Let X_1 be the number of rolls to get outcome 1, and similarly for X_2 and X_3. Let $X = \max(X_1, X_2, X_3)$. Then, by Example 8.19,

$$E[X] = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{p_1+p_2} - \frac{1}{p_1+p_3} - \frac{1}{p_2+p_3} + \frac{1}{p_1+p_2+p_3}$$
$$= 2 + 3 + 6 - (6/5 + 3/2 + 2) + 1 = 7.3.$$

There must be a way of getting this using first principles. Any suggestions?

2. (a) $P(X > Y) = 0$.
(b) $P(X \geq Y) = f(2,2) = 1/16$.
(c) $P(X + Y \text{is odd}) = f(1,2) + f(1,4) + f(2,3) = 1/8 + 1/4 + 1/8 = 1/2$.
(d) $P(X - Y \leq 1) = 1$.

14. (a) $1 = \sum_{(i,j)\neq(0,0)} \theta^{|i|}\theta^{|j|} = (1 + 2\theta/(1-\theta))^2 - 1$. This works for $\theta = (\sqrt{2}-1)/(\sqrt{2}+1) = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$.

(b) The $(0,0)$ term is 1, so this case is impossible unless $\theta = 0$ and we interpret $0^0 = 1$.

(c) $1 = \sum_{0 \le i < j} \theta^{i+j+2} = \theta^2 \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \theta^i \theta^j = \theta^2 \sum_{i=0}^{\infty} \theta^{2i+1}/(1-\theta) = \theta^3/[(1-\theta)^2(1+\theta)$. So the question is, does there exist a positive solution to $\theta^3 = (1-\theta)^2(1+\theta)$? This reduces to the quadratic $\theta^2 + \theta - 1 = 0$, so $\theta = (-1+\sqrt{5})/2$ works.

(d) $1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \theta^{i+j+1} = \theta/(1-\theta)^2$, so we need $(1-\theta)^2 = \theta$, and $\theta = (3-\sqrt{5})/2$ works.

(e) $1 = \sum_{j \ge 1} \sum_{i=1}^{c} (i^j - (i-1)^j)\alpha(\beta/c)^j = \sum_{j \ge 1} c^j \alpha(\beta/c)^j = \sum_{j \ge 1} \alpha\beta^j = \alpha\beta/(1-\beta)$, which requires $\alpha\beta/(1-\beta) = 1$.

(f) $1 = \sum_{1 \le i \le j} \alpha(i^n - (i-1)^n) j^{-n-2} = \alpha \sum_{j \ge 1} \sum_{i=1}^{j} (i^n - (i-1)^n) j^{-n-2} = \alpha \sum_{j \ge 1} j^n j^{-n-2} = \alpha \sum_{j \ge 1} j^{-2} = \alpha\pi^2/6$, so we need $\alpha = 6/\pi^2$. Here we are using the formula on page 23.

Independence holds only in case (c).

15. (a) By symmetry, both marginals are the same. At $i = 0$, we get $f(0) = \sum_{j \ne 0} \theta^{|j|} = 2\theta/(1-\theta)$. At $i \ne 0$, we get $f(i) = \theta^{|i|}(1 + 2\theta/(1-\theta))$.

(c) The marginal of X is $f_X(i) = \sum_{j=i+1}^{\infty} \theta^{i+j+2} = \theta^{2i+3}/(1-\theta)$ for $i \ge 0$. The marginal of Y is $f_Y(j) = \sum_{i=0}^{j-1} \theta^{i+j+2} = \theta^{j+2}(1-\theta^j)/(1-\theta)$ for $j \ge 1$.

(d) By symmetry, both marginals are the same. For $i \ge 0$, $f(i) = \sum_{j \ge 0} \theta^{i+j+1} = \theta^{i+1}/(1-\theta)$.

(e) The marginal of X is $f_X(i) = \sum_{j \ge 1} (i^j - (i-1)^j)\alpha(\beta/c)^j = \alpha(i\beta/c)/(1 - i\beta/c) - \alpha((i-1)\beta/c)/(1-(i-1)\beta/c)$ for $i = 1, 2, \ldots, c$. The marginal of Y is $f_Y(j) = \sum_{i=1}^{c} (i^j - (i-1)^j)\alpha(\beta/c)^j = \alpha\beta^j$ for $j \ge 1$.

(f) The marginal of X is $f_X(i) = \sum_{j \ge i} \alpha(i^n - (i-1)^n) j^{-n-2} = \alpha(i^n - (i-1)^n) \sum_{j \ge i} j^{-n-2}$ for $i \ge 1$, which cannot easily be simplified. The marginal of Y is $f_Y(j) = \sum_{i=1}^{j} \alpha(i^n - (i-1)^n) j^{-n-2} = \alpha j^{-2}$ for $j \ge 1$.