5010 solutions, Assignment 10. Chapter 6: 2, 3, 6, 10, 17.

2. (a) $G(s) = (s + s^2 + s^3 + \cdots + s^n)/n$, exists for all s. If $s \neq 1$, we can rewrite this as $G(s) = s(1 - s^n)/[n(1 - s)]$. Typo in book.

(b) $G(s) = (s^{-n} + s^{-n+1} + \cdots + s^{n-1} + s^n)/(2n + 1)$ exists for all $s \neq 0$. If also $s \neq 1$, this can be rewritten as $G(s) = s^{-n}(1 - s^{2n+1})/[(2n + 1)(1 - s)]$.

(c) $G(s) = \sum_{k=1}^{\infty}[k(k + 1)]^{-1}s^k$. The series converges if and only if $|s| \leq 1$. To evaluate the sum, $(sG(s))'' = \sum_{k=1}^{\infty} s^{k-1} = 1/(1 - s)$, hence $(sG(s))' = -\ln(1 - s) + C_0$, and $C_0 = 0$ follows by setting $s = 0$. Hence $sG(s) = (1 - s)\ln(1 - s) + s + C_1$, and $C_1 = 0$ follows by setting $s = 0$. We conclude that $G(s) = 1 + (1 - s)\ln(1 - s)/s$ This formula requires $s \neq 0$ and $s \neq 1$ as well.

(d) $G(s) = \sum_{k=1}^{\infty}[2k(k + 1)]^{-1}s^k + \sum_{k=-\infty}^{-1}[2k(k - 1)]^{-1}s^k$. The first series converges if and only if $|s| \leq 1$, and the second series converges if and only if $|s| \geq 1$, so we have convergence only at $|s| = 1$, which is not very useful.

(e) $G(s) = \sum_{k\in\mathbf{Z}}[(1-c)/(1+c)]c^{|k|}s^k$. The sum over $k \geq 0$ converges if and only if $|s| < 1/c$, and the sum over $k \leq 0$ converges if and only if $|s| > c$, so we require $c < |s| < 1/c$ for convergence. In this case we can write $G(s) = G_+(s) + G_-(s)$, where $G_+(s) = \sum_{k=0}^{\infty}[(1 - c)/(1 + c)](cs)^k = (1 - c)/[(1 + c)(1 - cs)]$ and $G_-(s) = \sum_{k=-\infty}^{-1}[(1 - c)/(1 + c)](s/c)^k = \sum_{k=1}^{\infty}[(1 - c)/(1 + c)](c/s)^k = [(1 - c)c/s]/[(1 + c)(1 - c/s)]$.

3. (a) Always by the theorem on page 242, since $e^{-\lambda(1-s)}$ is the Poisson pgf.

(b) No, because when the sine function is expanded in a power series about 0, the nonzero coefficients have alternating signs.

(c) Yes, shifted negative binomial pgf, assuming that r is a positive integer.

(d) Yes, binomial pgf, assuming that r is a positive integer.

(e) Yes. Expand in a powers series about 0, and check that all coefficients are nonnegative.

(f) Yes, if $-1 < \beta < 0$ and $\alpha = 1/\ln(1 + \beta)$. This will ensure that when expanded in a powers series about 0, all coefficients will be nonnegative.

6. If it were possible, we would have real polynomials of degree 5, $H_1(s)$ and $H_2(s)$, such that $sH_1(s)sH_2(s) = (s^2+s^3+\cdots+s^{12})/11 = s^2(1-s^{11})/[11(1-s)]$, or

$$11(1 - s)H_1(s)H_2(s) = 1 - s^{11}.$$

But $1 - s^{11}$ does not have such a factorization.

10. (a) X is binomial(n, p), so $G(s) = (q+ps)^n$. Next, $G'(s) = n(q+ps)^{n-1}p$, so $G''(s) = n(n-1)(q+ps)^{n-2}p^2$. Hence $E[X] = G'(1) = np$ and $E[X(X-1)] = G''(1) = n(n - 1)p^2$, and $\mathrm{Var}(X) = G''(1) + G'(1) - G'(1)^2 = np(1 - p)$.

(b) The probability that X is even is $(G(1) + G(-1))/2 = [1 + (q - p)^n]/2$. The reason that this works is that $[(1)^n + (-1)^n]/2 = 1$ if n is even and $= 0$ if n is odd.

(c) The same idea as in (b) should work here. Let 1, $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$ and $\bar{\omega}$ be the three cube roots of unity. Then $[(1)^n + \omega^n + \bar{\omega}^n]/3 = 1$ if n is divisible by 3 and $= 0$ if n is not divisible by 3 since $1 + \omega + \bar{\omega} = 0$. So the answer is $[G(1) + G(\omega) + G(\bar{\omega})]/3 = [1 + (q + p\omega)^n + (q + p\bar{\omega})^n]/3$.

17. Here X_n is the negative binomial distribution shifted to $\{0, 1, 2, 3 \ldots\}$, so its pgf is $[p/(1-qs)]^n$. Now we let $q = \lambda/n$ and let $n \to \infty$. We get

$$\left(\frac{1-\lambda/n}{1-\lambda s/n}\right)^n \to e^{-\lambda}/e^{-\lambda s} = e^{\lambda(s-1)}.$$