Math 5010 § 1. **First Midterm Exam** Name: Sample
Treibergs $\sigma\tau$ February 4, 2009

Questions from Mohammud Foondun's 5010 midterm exams from Fall 2008.

1. *You roll a pair of fair dice. What is the probability of the events:*
   (a) *The sum of the numbers is 2, 3, or 12?*
   (b) *The difference is even and the product is odd?*
2. *An urn contains 100 balls numbered 1 to 100. Four are removed at random without being replaced. Find the probability that the number on the last ball is smaller than the number on the first ball.*
3. *Find the probability that in 24 throws of a pair of dice, double sixes fails to appear.*
4. *A fair coin is tossed four times. What is the probability of*
   (a) *At least three heads?*
   (b) *A run of three or more consecutive heads?*
5. *If a coin is tossed five times, show that the probability that it shows heads on an odd number of tosses is* $\frac{1}{2}$.
6. *Use the Binomial Theorem to show the following*
   (a) $\sum_{i=0}^{n} \binom{n}{i}(-1)^i = 0.$
   (b) $\sum_{k=1}^{n} \binom{n}{k} k = n2^{n-1}.$
7. (a) *How many different linear arrangements are there of the letters A, B, C, D, E and F for which A and B are next to each other and C and D are also next to each other?*
   (b) *A committee of five is to be chosen from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?*
8. *Suppose that each of the 3 men at a party throws his hat into the center of the room. The hats are mixed up and then each man randomly selects a hat. What is the probability that none of the men selects his own hat? What would the probability be if there were 4 men instead of 3?*
9. *A laboratory blood test is 95% effective in detecting a certain disease when it is in fact present. However, the test also yields a "false positive" result for 1% of the healthy people tested. In other words, if a healthy person is tested, then, with probability 0.01, the test indicates that he has the disease. If 0.5% of the population actually have the disease, what is the probability that the person has the disease given that the test result is positive?*
10. *Suppose that we have three cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The three cards are mixed up in the hat and one card is randomly selected and put down on the floor. If the upper side of the chosen card is red, what is the probability that the other side is also colored red?*

Solved problems from my Math 3070 - 1 exam, given Jan. 30, 2008.

1. *A truth serum given to a suspect is known to be 90% reliable when the person is guilty and 99% reliable when the person is innocent. In other word, 10% of the guilty are judged innocent while 1% of the innocent are judged guilty. A suspect is selected from a group of suspects of which only 5% are guilty of ever committing a crime. Given that the serum indicates that he is guilty, what is the probability that he is innocent?*

Let A be the event that the suspect is guilty of the crime. We are given $P(A) = .05$ so $P(A') = .95$. Let B be the event that the serum reveals that the suspect is guilty. We are given that $P(B|A) = .90$ so $P(B'|A) = .10$ and that $P(B'|A') = .99$ so $P(B|A') = .01$. The total probability formula says $P(B) = P(A' \cap B) + P(A \cap B) = P(A')P(B|A') + P(A)P(B|A) = (.95)(.01) + (.05)(.90) = .0545$. We are asked to compute

$$P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(A')P(B|A')}{P(B)} = \frac{(.95)(.01)}{.0545} = \boxed{.174}.$$

2. *One hundred Salt Sake City Democrats were were asked their opinions of two candidates B and H, running in the presidential primary. Of these, 65 said they liked B, 55 said they liked H and 25 said they liked both. What is the probability that someone likes at least one? Given that someone doesn't like H, what is the probability that they likes B?*

Let B be the event that B is liked and H be the event that H is liked and $B \cap H$ the event that both are liked. We are given $P(B) = .65$ and $P(H) = .55$ and $P(B \cap H) = .25$. The event that someone likes at least one is $B \cup H$. The union formula says $P(B \cup H) = P(B) + P(H) - P(B \cap H) = .65 + .55 - .25 = \boxed{.95}$. The probability that someone likes B given that they don't like H is

$$P(B|H') = \frac{P(B \cap H')}{P(H')} = \frac{P(B) - P(B \cap H)}{1 - P(H)} = \frac{.65 - .25}{1 - .55} = \boxed{.889}.$$

3. *A standard deck of 52 cards consists of four suits* $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. *Each suit has 13 different kinds of cards* $\{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$. *Suppose that two cards are randomly drawn from the deck without replacement. What is the probability that the two cards are of the same suit or of the same kind?*

Let A_i be the event that both cards have the same suit $i \in \{1, 2, 3, 4\}$. Let B_j be the event that both cards have the same kind $j \in \{1, \ldots, 13\}$. There are $\binom{13}{2}$ ways to pick two cards of suit i. There are $\binom{4}{2}$ ways to choose two cards of kind j. Observe that the events $A_1, \ldots, A_4$, $B_1, \ldots, B_{13}$ are mutually exclusive. Since, for example, if two cards are of the same suit they cannot be of the same kind. Thus, assuming that all draws are equally likely, the probability of drawing the same suit or the same kind is

$$P(A_1 \cup \cdots \cup A_4 \cup B_1 \cup \cdots \cup B_{13}) = \frac{4\binom{13}{2} + 13\binom{4}{2}}{\binom{52}{2}} = \frac{4\frac{13\bullet 12}{2} + 13\frac{4\bullet 3}{2}}{\frac{52\bullet 51}{2}} = \frac{15}{51} = \boxed{.294}.$$

4.*A bag contains 26 scrabble tiles, each labeled by a different letter of the alphabet. Five tiles are randomly selected in order from the bag without replacement. How many different five letter words can be selected from the bag? Let A be the event that the word is in alphabetical order. Find the probability P(A). Let B be the event that the word chosen is "FIRST." Are the events A and B independent? Why?*

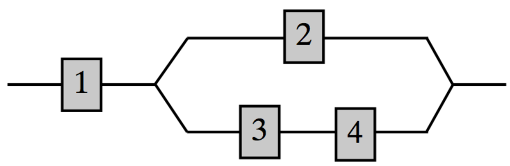
We are counting the number of permutations of 26 letters taken 5 at a time (order is important!) so the number of five letter words drawn is $P_{5,26} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \boxed{7893600}$. Given five different letters, there are $5! = 120$ ways to order them, and only one ordering is in alphabetical order. Thus there are only $P_{5,25}/5!$ ways to choose words in alphabetical order. The probability

$$P(\text{word in alphabetical order}) = \frac{P_{5,26}/5!}{P_{5,26}} = \frac{1}{120} = \boxed{.00833}.$$

The probability that "FIRST" is chosen is $P(B) = 1/P_{5,26} = 1.267\times 10^{-7}$. Observe that "FIRST" is in alphabetical order so that $B \subset A$ and $A \cap B = B$. The events would be independent if $P(A \cap B) = P(A)P(B)$. Thus A and B are $\boxed{\text{not independent}}$ because

$$P(A)P(B) = (.00833)(1.267 \times 10^{-7}) \neq P(A \cap B) = P(B) = 1.267 \times 10^{-7}.$$

5. *Consider the system of components connected as in the diagram. If a subsystem consists of two units connected in parallel, then the subsystem works if and only if either one of the units work. If a subsystem consists of two units connected in series, then the subsystem works if and only if both of the units work. In other words, the system works if and only if you can trace a path through the network from left to right that passes only through working components. Assume that the components work independently of one another and that P(component works) $= .8$, calculate P(system works).*



Let A_i be the event that the ith component works. Since the events are mutually independent, the probability of intersection can be gotten by multiplying probabilities. Thus, using the formula for union,

$$\begin{aligned}
P(\text{system works}) &= P\Big(A_1 \cap \big(A_2 \cup (A_3 \cap A_4)\big)\Big) \\
&= P(A_1)\, P\big(A_2 \cup (A_3 \cap A_4)\big) \\
&= P(A_1)\, \big(P(A_2) + P(A_3 \cap A_4) - P(A_2 \cap A_3 \cap A_4)\big) \\
&= P(A_1)\, \big(P(A_2) + P(A_3)P(A_4) - P(A_2)P(A_3)P(A_4)\big) \\
&= (.8)\big(.8 + (.8)(.8) - (.8)(.8)(.8)\big) \\
&= \boxed{.7424}.
\end{aligned}$$