

Questions from Mohammad Foondun's 5010 midterm exams from Fall 2008.

1. Find the value of c which makes the following function a probability mass function
 - (a) $f(x) = \frac{c 2^x}{x!}$ for $x \in \{0, 1, 2, \dots\}$?
 - (b) $f(x) = \frac{c}{x(x+1)}$ for $x \in \{1, 2, 3, \dots\}$?
2. Let X be a random variable uniformly distributed on $1 \leq x \leq m$. Calculate $\mathbf{P}(X = k \mid a \leq X \leq b)$ and $\mathbf{P}(X > n + k \mid X > n)$.
3. Let X be a binomial random variable with parameters n and p . Show that $\mathbf{E}(X) = np$. Fix $\mathbf{E}(X)$ and show that as $n \rightarrow \infty$, $\mathbf{P}(X = 0) \rightarrow \exp(-\mathbf{E}(X))$.
4. Let X be a binomial random variable as defined in the previous problem. Define A to be the event that $X \neq 0$. Find $\mathbf{E}(X \mid A)$.

Questions from Stewart Ethier's 5010 midterm exam from Fall 2007.

1. For each random variable described, choose the distribution that best approximates it. (Alternatively, choose two distributions; if either is correct, you'll get half credit.) Each of the choices below is an answer exactly once. You may write directly on this sheet. Hint: You don't have to guess. Think about each of the random variables (a)-(h). What is its range of possible values? Can it be expressed in terms of Bernoulli trials? Then try to match it with one of 1-8. Warning: If you guess, your expected score is 1 out of 8. So skip this one if you are guessing.
 1. Bernoulli (0 or 1 valued)
 2. binomial
 3. geometric
 4. hypergeometric
 5. negative binomial
 6. Poisson
 7. discrete uniform
 8. multinomial
 - (a) The number of matches between the numbers chosen in a lottery and your pre-selected numbers.
 - (b) The number of wives in a random nonpolygamous Utah family.
 - (c) The numbers of Republicans, Democrats, and Independents in a random sample of 1,000 voters.
 - (d) The number of customers using the bookstore ATM between 12:00 and 12:15.
 - (e) The number of attempts an extra-point kicker needs to score 10 points.
 - (f) The three-digit number on a random Utah license plate.
 - (g) The number of students in a class of 25 having blood type A+.
 - (h) The number of rolls of a pair of dice needed to achieve a total of 7.

2. In a sequence of independent Bernoulli trials, each with success probability $1/3$, find:
 - (a) the probability of exactly 6 successes in the first 13 trials.
 - (b) the probability that the first success occurs on or after the 6th trial.
 - (c) the probability that the 6th success occurs on the 13th trial.
3. One of the numbers 1 through 8 is randomly chosen. You are to try to guess the number chosen by asking questions with yes or no answers. Compute the expected number of questions you will need to ask in each of the two cases:
 - (a) Your i th question is to be: Is it i ?, $i = 1, 2, 3, 4, 5, 6, 7, 8$, in that order.
 - (b) With each question you eliminate half of the remaining numbers.
4. The joint probability mass function of the random variables X, Y, Z satisfies $f(1, 2, 3) = f(2, 1, 1) = f(2, 2, 1) = f(2, 3, 2) = 1/4$, with $f(i, j, k) = 0$ for all other (i, j, k) .
 - (a) Find the marginal probability mass functions of X, Y , and Z .
 - (b) Find $\mathbf{E}(XYZ)$.
 - (c) Find $\mathbf{E}(X | Y)$.
5. Suppose you are faced with a matching problem on an exam. There are 10 questions numbered 1 through 10, and 10 answers labeled (a) through (j) , each of which is the correct answer exactly once. You must match a letter to each number. Suppose you don't have a clue as to the correct answers, so you decide to randomly guess, and you decide to use each letter once and only once. Let X be the number of questions answered correctly. Writing X as the sum $X_1 + X_2 + \cdots + X_{10}$, with X_i being the random variable that is 1 if question i is answered correctly, and 0 otherwise. Show $\mathbf{E}[X_i] = 1$ for each i , hence $\mathbf{E}[X] = 10$. Find $\mathbf{Var}(X)$ using the same technique.

Questions from Stewart Ethier's 5010 midterm exam from Fall 2008.

1. A person tosses a fair coin until a head appears for the first time. If the first head appears on the n th flip, the person wins 2^n dollars. Let X denote the player's winnings.
 - (a) Show that $\mathbf{E}[X] = \infty$. (Thus, a fair entry fee is infinite! This is the St. Petersburg paradox.)
 - (b) Find $\mathbf{E}[\sqrt{X}]$. (This is the amount you should be willing to pay to play this game if your utility function is the square root function.) Use $1 + a + a^2 + a^3 + \cdots = 1/(1 - a)$ if $|a| < 1$.
2. Let X have the geometric distribution $\mathbf{P}(X = k) = q^{k-1}p$, where $k = 1, 2, 3, \dots$ and $q = 1 - p$. We gave three derivations of $\mathbf{E}[X]$ in class. Provide one of them.
3. An urn contains 3 balls labeled 1, 2, and 3. Two balls are removed without replacement. Let X be the minimum of the numbers on the two balls, Y the maximum.
 - (a) Give the joint probability mass function of X and Y . (Fill in a 3×3 or 2×2 table.)
 - (b) Find $\mathbf{Cov}(X, Y)$.
4. A fair coin is tossed 3 times and a die (with faces numbered 1, 1, 2, 2, 3, 3) is rolled once. Let X be the number of heads and let Y be the number appearing on upper face of the die.
 - (a) Find the joint distributions of X and Y .

- (b) Find $\mathbf{P}(X + Y = 4)$.
5. A deck contains 52 cards. Assume that a 5-card hand is dealt (without replacement of course). Let X be the number of aces in the hand. Notice that $X = X_1 + \cdots + X_5$, with X_i being the indicator of the event that the i th card dealt is an ace.
- (a) Find $\mathbf{E}[X]$.
- (b) Find $\mathbf{Var}(X)$.
6. You have 13 cards, labeled 1–13 for simplicity. They are shuffled and then turned up one by one. Let X be the number of integers n ($1 \leq n \leq 13$) such that card n is the n th card exposed.
- (a) What are the possible values of the random variable X ? (Be careful. This is not quite as simple as it first appears.)
- (b) Write X as the sum $\mathbb{I}_{A_1} + \mathbb{I}_{A_2} + \cdots + \mathbb{I}_{A_{13}}$, with A_n being the event that card n is the n th card exposed. Find $\mathbf{P}(A_n)$.
- (c) Use part (b) to find $\mathbf{E}[X]$.

Some Extra Sample Problems.

1. Suppose that there are a amber and b beryl balls in an urn. Suppose that $k \leq a + b$ are randomly withdrawn without replacement. Let X be the number of amber balls withdrawn. Using the method of indicators or otherwise, find $\mathbf{E}(X)$ and $\mathbf{Var}(X)$.
2. Let X and Y be independent random variables, such that X is uniform on $\{1, 2, 3, \dots, m\}$ and Y is uniform on $\{1, 2, 3, \dots, n\}$ where $1 \leq m \leq n$. Let $Z = X + Y$. Find the pmf $f_Z(z)$, $\mathbf{E}(Z)$ and $\mathbf{Var}(Z)$.
3. Suppose that there are n dice whose six sides instead of being numbered are labeled by an apple on one side, a banana on two sides and a cherry on three sides. Suppose all n dice are rolled. Let X be the number of apples showing, Y the number of bananas and Z the number of cherries. Find the joint probability $f(x, y, z)$. Find the marginal probabilities $f_{XY}(x, y)$ and $f_X(x)$.
4. Let X and Y be random variables whose joint probability is $f(1, 1) = f(3, 2) = .2$, $f(1, 2) = .5$, $f(2, 2) = .1$ and $f(i, j) = 0$ for other (i, j) . Find $f_{X|Y}(x | y)$ and $\mathbf{E}(X | Y)$.
5. Suppose that $\mathbf{E}(|X|^\alpha) = 0$ for some $\alpha > 0$. Show that $\mathbf{P}(X = 0) = 1$.
6. Let X_n be a sequence of independent Bernoulli trials ($X_n \in \{0, 1\}$) such that $\mathbf{P}(X_n = 1) = p$ for all n . Show that the sample means converge in probability as $n \rightarrow \infty$,

$$\frac{1}{n} (X_1 + \cdots + X_n) \xrightarrow{P} p.$$

7. Urn 1 contains 10 white and 3 black balls; urn 2 contains 3 white and 5 black balls. Two randomly selected balls are transferred from No. 1 and placed in No. 2 and then 4 balls are taken from the latter. Let the rv X be the number of white balls chosen. Find $f_X(x)$.
8. Roll a die. Let X be the number showing. Then flip X fair coins. Let Y be the number of heads. Find $\mathbf{E}(Y)$.