Math 5010-2 Name:
Midterm Exam 2 — with solutions
11/28/07

1. For each random variable described, choose the distribution that best approximates it. (Alternatively, choose two distributions; if either is correct, you'll get half credit.) Each of the choices below is an answer exactly once. You may write directly on this sheet. Hint: You don't have to guess. Think about each of the random variables (a)–(h). What is its range of possible values? Can it be expressed in terms of Bernoulli trials? Then try to match it with one of 1–8. Warning: If you guess, your expected score is 1 out of 8! So skip this one if you are guessing.

1. Bernoulli (0 or 1 valued)
2. binomial
3. geometric
4. hypergeometric
5. negative binomial
6. Poisson
7. discrete uniform
8. multinomial

(a) The number of matches between the numbers chosen in a lottery and your pre-selected numbers. 4, hypergeometric
(b) The number of wives in a random nonpolygamous Utah family. 1, Bernoulli
(c) The numbers of Republicans, Democrats, and Independents in a random sample of 1,000 voters. 8, multinomial
(d) The number of customers using the bookstore ATM between 12:00 and 12:15. 6, Poisson
(e) The number of attempts an extra-point kicker needs to score 10 points. 5, negative binomial
(f) The three-digit number on a random Utah license plate. 7, discrete uniform.
(g) The number of students in a class of 25 having blood type A+. 2, binomial.
(h) The number of rolls of a pair of dice needed to achieve a total of 7. 3, geometric

2. In a sequence of independent Bernoulli trials, each with success probability 1/3, find:
(a) the probability of exactly 6 successes in the first 13 trials.
Binomial(13,1/3): $\binom{13}{6}(1/3)^6(2/3)^7$.
(b) the probability that the first success occurs on or after the 6th trial.
Equivalent to failures at the first 5 trials: $(2/3)^5$.
(c) the probability that the 6th success occurs on the 13th trial.
Negative binomial$(6, 1/3)$: $\binom{12}{5}(1/3)^6(2/3)^7$.

3. One of the numbers 1 through 8 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes" or "no" answers. Compute the expected number of questions you will need to ask in each of the two cases:
(a) Your ith question is to be: "Is it i?", $i = 1, 2, 3, 4, 5, 6, 7, 8$, in that order.
It is i with probability 1/8 for each i. You ask until you get a "yes" or until the 7th "no." So the r.v. assumes values $1, 2, 3, 4, 5, 6, 7, 7$ with probability 1/8 each. The mean is $(1+2+3+4+5+6+7+7)/8 = 35/8 = 4.375$.
(b) With each question you eliminate half of the remaining numbers.
After the first question, there are four possibilities. After the second there are two. After the third there is one. So you will ask exactly 3 questions. There the mean number is 3.

4. The joint probability mass function of the random variables X, Y, Z satisfies $f(1,2,3) = f(2,1,1) = f(2,2,1) = f(2,3,2) = 1/4$, with $f(i,j,k) = 0$ for all other (i,j,k).

(a) Find the marginal probability mass functions of X, Y, and Z.

For X, it is 1 with prob. 1/4 and 2 with prob. 3/4 (this is because there are 3 terms with a 2 in the x variable with each having prob. 1/4). For Y, it is 1 with prob. 1/4, 2 with prob. 1/2, and 3 with prob. 1/4. For Z, it is 1 with prob. 1/2, 2 with prob. 1/4, and 3 with prob. 1/4.

(b) Find $E[XYZ]$.

$E[XYZ] = (1 \cdot 2 \cdot 3)/4 + (2 \cdot 1 \cdot 1)/4 + (2 \cdot 2 \cdot 1)/4 + (2 \cdot 3 \cdot 2)/4 = (6 + 2 + 4 + 12)/4 = 6.$

5. Recall Quiz 9: "Suppose you are faced with a matching problem on an exam. There are 10 questions numbered 1 through 10, and 10 answers labeled (a) through (j), each of which is the correct answer exactly once. You must match a letter to each number. Suppose you don't have a clue as to the correct answers, so you decide to randomly guess, and you decide to use each letter once and only once. Let X be the number of questions answered correctly."

Writing X as the sum $X_1 + X_2 + \cdots + X_{10}$, with X_i being the random variable that is 1 if question i is answered correctly, and 0 otherwise, we saw that $E[X_i] = 1/10$ for each i, hence $E[X] = 1$. Find $\mathrm{Var}(X)$ using the same technique.

$\mathrm{Var}(X) = \sum_i \mathrm{Var}(X_i) + 2\sum\sum_{1 \le i < j \le 10} \mathrm{Cov}(X_i, X_j) = 10\mathrm{Var}(X_1) + 90\mathrm{Cov}(X_1, X_2) = 10(1/10)(1 - 1/10) + 90(1/90 - (1/10)^2) = 1$. Here we used $\mathrm{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2]$, and $E[X_1X_2] = P(\text{questions 1 and 2 correct}) = 8!/10! = 1/(10 \cdot 9) = 1/90$.

6. Roll a fair die. If number n appears $(1 \le n \le 6)$, toss n coins. Let X be the number of heads that result.

(a) Find the pgf of X. Hint: It should be the composition of two pgfs.

Let N be the outcome of the die, which has pgf $G(s) = (s + s^2 + s^3 + s^4 + s^5 + s^6)/6$. Let X_1 be the indicator of a head on a single toss, which has pgf $(1+s)/2$. X has pgf equal to the composition of the two, namely $G(H(s)) = ((1+s)/2 + [(1+s)/2]^2 + [(1+s)/2]^3 + [(1+s)/2]^4 + [(1+s)/2]^5 + [(1+s)/2]^6)/6$.

(b) Use part (a) to find $P(X = 3)$.

It will be the coefficient of s^3 when we write the pgf $G(H(s))$ as a polynomial. This will be $(1/8 + 4/16 + 10/32 + 20/64)/6 = 1/6$.