Math 5010-2
Midterm Exam 2
November 19, 2008

1. You have 13 cards, labeled 1–13 for simplicity. They are shuffled and then turned up one by one. Let X be the number of integers n ($1 \leq n \leq 13$) such that card n is the nth card exposed.

(a) What are the possible values of the random variable X? (Be careful. This is not quite as simple as it first appears.)

Sol. The values are the integers 0–13, with one exception, and this is the tricky part. If there are 12 matches, then there must actually be 13, so 12 is impossible. So the answer is 0–11 and 13.

(b) Write X as the sum $1_{A_1} + 1_{A_2} + \cdots + 1_{A_{13}}$, with A_n being the event that card n is the nth card exposed. Find $P(A_n)$.

Sol. $P(A_n) = 12!/13! = 1/13$. The point is that there are 13! ways to order the cards. If card i is in position i, there are only 12! ways to order the remaining cards.

(c) Use part (b) to find $E[X]$.

Sol. $E[X] = E[1_{A_1} + 1_{A_2} + \cdots + 1_{A_{13}}] = E[1_{A_1}] + E[1_{A_2}] + \cdots + E[1_{A_{13}}] = 13E[1_{A_1}] = 13P(A_1) = 13/13 = 1$.

2. A person tosses a fair coin until a head appears for the first time. If the first head appears on the nth flip, the person wins 2^n dollars. Let X denote the player's winnings.

(a) Show that $E[X] = \infty$. (Thus, a "fair" entry fee is infinite! This is the St. Petersburg paradox.)

(b) Find $E[\sqrt{X}]$. (This is the amount you should be willing to pay to play this game if your "utility function" is the square root function.) Use $1 + a + a^2 + a^3 + \cdots = 1/(1-a)$ if $|a| < 1$.

The payoff is 2^n if $n-1$ heads are followed by a tail, hence with probability 2^{-n}. Therefore,

$$E[X] = \sum_{n=1}^{\infty} 2^n 2^{-n} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty.$$

and

$$E[\sqrt{X}] = \sum_{n=1}^{\infty} \sqrt{2^n} 2^{-n} = \sum_{n=1}^{\infty} 2^{-n/2} = \frac{2^{-1/2}}{1 - 2^{-1/2}} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1.$$

3. Let X have the geometric distribution $P(X = k) = q^{k-1}p$, where $k = 1, 2, 3, \ldots$ and $q = 1 - p$. We gave three derivations of $E[X]$ in class. Provide one of them.

Sol. Method 1. $\mu = \sum_1^\infty nq^{n-1}p = p(d/dq)\sum_1^\infty q^n = p(d/dq)[q/(1-q)] = p/(1-q)^2 = p/p^2 = 1/p$.

Method 2. $\mu = \sum_1^\infty P(X \geq n) = \sum_1^\infty q^{n-1} = 1/(1-q) = 1/p$.

Method 3. By conditioning on the result of the first trial, we get $\mu = p \cdot 1 + (1-p)(1+\mu)$, and solving gives $\mu = 1/p$.

4. An urn contains 3 balls labeled 1, 2, and 3. Two balls are removed without replacement. Let X be the minimum of the numbers on the two balls, Y the maximum.

(a) Give the joint probability mass function of X and Y. (Fill in a 3×3 or 2×2 table.)

(b) Find $\text{Cov}(X, Y)$.

Sol.

$$P\{(X,Y) = (2,1)\} = P\{(X,Y) = (3,1)\} = P\{(X,Y) = (3,2)\} = \frac{1}{3}$$

so $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = (2+3+6)/3 - [(2+3+3)/3][(1+1+2)/3] = 11/3 - (8/3)(4/3) = (33-32)/9 = 1/9$.

5. A fair coin is tossed 3 times and a die (with faces numbered 1, 1, 2, 2, 3, 3) is rolled once. Let X be the number of heads and let Y be the number appearing on upper face of the die.

(a) Find the distributions of X and Y.
(b) Find $P(X+Y=4)$.

Sol. (a) X equals $0, 1, 2, 3$ with probabilities $1/8, 3/8, 3/8, 1/8$. Y equals $1, 2, 3$ with probabilities $1/3, 1/3, 1/3$.
(b) $P(X+Y=4) = P((X,Y) = (1,2), (2,2), (3,1)) = (3/8)(1/3)+(3/8)(1/3)+(1/8)(1/3) = 7/24$.

6. A deck contains 52 cards. Assume that a 5-card hand is dealt (without replacement of course). Let X be the number of aces in the hand. Notice that $X = X_1 + \cdots + X_5$, with X_i being the indicator of the event that the ith card dealt is an ace.
(a) Find $E[X]$.
(b) Find $\mathrm{Var}(X)$.

Sol. (a) $P(A_i) = 4/52$ so $E[X] = 5(4/52) = 5/13$.
(b) $\mathrm{Var}(X) = 5\mathrm{Var}(X_1) + 5 \cdot 4\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Var}(X_1) = (1/13)(1 - 1/13)$, and $\mathrm{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] = (4/52)(3/51) - (4/52)^2 = (1/13)(1/17 - 1/13)$. Result is $5(1/13)(1 - 1/13) + 20(1/13)(1/17 - 1/13) = 940/2873 \approx 0.327184$.
The above is the recommended solution. Alternatively, we could use the hypergeometric distribution. Then

$$P(X=k) = \binom{4}{k}\binom{48}{5-k}\bigg/\binom{52}{5}.$$

Hence

$$E[X] = 1\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} + 2\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} + 3\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} + 4\frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}.$$

and $\mathrm{Var}(X) = E[X(X-1)] + E[X] - (E[X])^2$, where

$$E[X(X-1)] = (2)(1)\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} + (3)(2)\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} + (4)(3)\frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}.$$

With sufficient patience, one can evaluate these formulas numerically.