334[25] *What is the moment generating function of the two-sided exponential density? Where is it defined?*

The two sided exponential density has two parameters, $0 < p < 1$ and $\lambda > 0$. Its density distribution function is

$$f_X(x) = \begin{cases} \lambda p e^{\lambda x}, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ \lambda(1-p)e^{-\lambda x}, & \text{if } x > 0. \end{cases}$$

The moment generating function is

$$\begin{aligned} M_X(t) = \mathbf{E}(e^{tX}) &= \int_{x=-\infty}^{\infty} e^{tx}\, f_X(x)\, dx \\ &= \lambda p \int_{x=-\infty}^{0} e^{tx}\, e^{\lambda x}\, dx + \lambda(1-p)\int_{x=0}^{\infty} e^{tx}\, e^{-\lambda x}\, dx \\ &= \lambda p \int_{x=-\infty}^{0} e^{(t+\lambda)x}\, dx + \lambda(1-p)\int_{x=0}^{\infty} e^{(t-\lambda)x}\, dx \\ &= \lambda p \left(\frac{e^{(t+\lambda)x}}{t+\lambda} \right)\Bigg|_{x=-\infty}^{0} + \lambda(1-p)\left(\frac{e^{(t-\lambda)x}}{t-\lambda} \right)\Bigg|_{x=0}^{\infty} \\ &= \frac{\lambda p}{\lambda + t} + \frac{\lambda(1-p)}{\lambda - t} = \frac{\lambda^2 + \lambda(1-2p)t}{\lambda^2 - t^2}. \end{aligned}$$

For the first integral to be finite requires $t + \lambda > 0$. For the second integral to be finite requires $t - \lambda < 0$. Together, this requires $|t| < \lambda$.

391[7] *Let U and V be independently and uniformly distributed on $(0,1)$. Find the joint density of*

$$X = \frac{\sqrt{U}}{\sqrt{U} + \sqrt{V}}, \qquad Y = \sqrt{U} + \sqrt{V}.$$

By considering $\mathbf{P}(X \le x \mid Y \le 1)$, devise a rejection sampling procedure for simulating a random variable with density $6x(1-x)$ on $(0,1)$.

Since U and V are independent and uniform on $(0,1)$, it follows that the joint density is $f_{UV}(u,v) = f_U(u)f_V(v) = 1$ if $0 < u, v < 1$ and zero otherwise. The mapping $x(u,v) = u^{1/2}/(v^{1/2} + v^{1/2})$ and $y(u,v) = u^{1/2} + v^{1/2}$ is a one-to-one mapping from the unit square $0 < u, v < 1$ to the region

$$\mathcal{G} = \left\{ (x,y) :\ 0 < x < 1 \text{ and } 0 < y < \min\left(\frac{1}{x}, \frac{1}{1-x} \right) \right\}.$$

This is because the mapping has an inverse $u = x^2y^2$ and $v = (1-x)^2y^2$. The Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 2xy^2 \cdot 2(1-x)^2 y - 2x^2 y \cdot \left(-2(1-x)y^2\right) = 4x(1-x)y^3.$$

The density in the new variables is given by the change of variables formula

$$f_{X,Y}(x,y) = f_{U,V}\big(u(x,y), v(x,y)\big) \left|\frac{\partial(u,v)}{\partial(x,y)}\right|.$$
$$= 4x(1-x)y^3.$$

Note that $(0,x) \times (0,y) \subset \mathcal{G}$ for $0 < x, y < 1$. Thus for $0 < x, y < 1$,

$$\mathbf{P}(X \le x \text{ and } Y \le y) = \int_{(0,x)\times(0,y)} f_{XY}(x,y)\,dx\,dy = 4\int_0^y \int_0^x x(1-x)y^3\,dx\,dy$$
$$= \frac{1}{2}x^2y^4 - \frac{1}{3}x^3y^4;$$
$$\mathbf{P}(Y \le 1) = \mathbf{P}(X \le 1 \text{ and } Y \le 1) = \frac{1}{6}.$$

Rejection sampling is a way to produce random variables with one distribution from variables of another. Here, we start with two independent, uniform on $(0,1)$ random numbers U, V as might come from a random number generator on a computer, compute X and Y, and then output X if $Y \le 1$ or go back and do it again if $Y > 1$. The resulting cumulative distribution of the output is given for $0 < x < 1$ by

$$F(x) = \mathbf{P}(X \le x \mid Y \le 1) = \frac{\mathbf{P}(X \le x \text{ and } Y \le 1)}{\mathbf{P}(Y \le 1)} = 3x^2 - 2x^3.$$

It follows, that the output variable has density

$$f(x) = \frac{dF}{dx} = 6x(1-x).$$

A*. *Suppose $X_1, X_2, X_3, \ldots$ is a sequence of independent random variables all Poisson distributed with parameter $\lambda = 2$. Let*

$$Y_n = \frac{S_n - \mathbf{E}(S_n)}{\sqrt{\mathbf{Var}(S_n)}}$$

where $S_n = X_1 + X_2 + \cdots + X_n$. Show that

$$Y_n \xrightarrow{D} Z \qquad \text{as} \qquad n \to \infty$$

converges in distribution, where $Z \sim N(0,1)$ is the standard normal variable. (Don't quote CLT.) Then find $\mathbf{P}(18 \le S_{10} \le 20)$ approximately.

The method is the same as proving the de Moivre-Laplace Theorem, or the Central Limit Theorem. By the Continuity Theorem, it suffices to show that for some $b > 0$, the moment generating functions are all finite for $|t| < b$ and converge: for all $|t| \le b/2$,

$$M_{X_n}(t) \to M_Z(t) \qquad \text{as} \qquad n \to \infty.$$

We know that $\mathbf{E}(X_n) = \lambda$ so that $\mathbf{E}(S_n) = n\lambda$. Also $\mathbf{Var}(X_n) = \lambda$ so that by independence, $\mathbf{Var}(S_n) = n\lambda$. Hence

$$Y_n = \frac{S_n}{\sqrt{n\lambda}} - \sqrt{n\lambda}$$

so that, using the probability generating function for the Poisson rv, $\mathbf{E}(s^{X_n}) = e^{\lambda(s-1)}$, the moment generating function for Y_n is

$$\begin{aligned}
M_{Y_n}(t) &= \mathbf{E}\left(e^{tY_n}\right) \\
&= \mathbf{E}\left(\exp\left(\frac{tS_n}{\sqrt{n\lambda}} - t\sqrt{n\lambda}\right)\right) \\
&= \exp\left(-t\sqrt{n\lambda}\right)\mathbf{E}\left(\exp\left(\frac{t}{\sqrt{n\lambda}}\right)^{S_n}\right) \\
&= \exp\left(-t\sqrt{n\lambda}\right)\exp\left(\lambda\left[\exp\left(\frac{t}{\sqrt{n\lambda}}\right) - 1\right]\right)^n \\
&= \exp\left(\lambda n\left[\exp\left(\frac{t}{\sqrt{n\lambda}}\right) - 1\right] - t\sqrt{n\lambda}\right)
\end{aligned}$$

since the X_n are independent. Taking logarithms, and expanding the exponential we see that for, say $|t| \leq 1$, as $n \to \infty$,

$$\begin{aligned}
\log M_{Y_n}(t) &= \lambda n\left[\exp\left(\frac{t}{\sqrt{n\lambda}}\right) - 1\right] - t\sqrt{n\lambda} \\
&= \lambda n\left[\frac{t}{\sqrt{n\lambda}} + \frac{t^2}{2n\lambda} + \mathbf{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)\right] - t\sqrt{n\lambda} \\
&= \frac{t^2}{2} + \mathbf{O}\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

It follows from the continuity of the exponential that for all $|t| \leq 1$,

$$\lim_{n\to\infty} M_{Y_n}(t) = e^{\frac{1}{2}t^2} = M_Z(t)$$

which completes the argument.

Finally, let's approximate Y_{10} by Z to solve the last question. Using $\lambda = 2$, $\mathbf{E}(S_{10}) = 20$ and $\mathbf{Var}(S_{10}) = 20$ and by standardizing,

$$\begin{aligned}
\mathbf{P}(18 \leq S_{10} \leq 20) &= \mathbf{P}\left(\frac{18-20}{\sqrt{20}} \leq \frac{S_{10} - \mathbf{E}(S_{10})}{\sqrt{\mathbf{Var}(S_{10})}} \leq \frac{20-20}{\sqrt{20}}\right) \\
&= \mathbf{P}\left(-\frac{1}{\sqrt{5}} \leq Y_{10} \leq 0\right) \\
&\approx \mathbf{P}\left(-\frac{1}{\sqrt{5}} \leq Z \leq 0\right) \\
&= \Phi(0) - \Phi\left(-\frac{1}{\sqrt{5}}\right) \approx \Phi(0) - \Phi\left(-0.4472\right) \\
&\approx 0.5000 - 0.3272 = 0.1727.
\end{aligned}$$