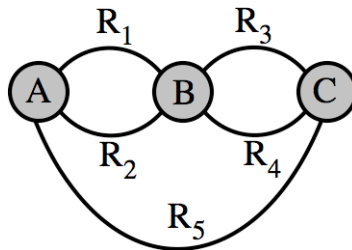


76[12] Two roads join Ayton to Beaton, and two further roads join Beaton to the City. Ayton is directly connected to the City by railway. All four roads and the railway are each independently blocked by snow with probability p . I am at Ayton.

- (a) Find the probability that I can drive to the City.
 (b) Find the probability that I can travel to the City.
 (c) Given that I can travel to the City, what is the probability that the railway is blocked?



Roads connecting three cities.

(a.) Let S_i be the event that the i th road is blocked. The event AB that I can drive from A to B is that either of the roads R_1 or R_2 are open, which is $AB = S_1^c \cup S_2^c$. Using independence,

$$\mathbf{P}(AB) = \mathbf{P}(S_1^c \cup S_2^c) = \mathbf{P}((S_1 \cap S_2)^c) = 1 - \mathbf{P}(S_1 \cap S_2) = 1 - \mathbf{P}(S_1) \mathbf{P}(S_2) = 1 - p^2.$$

Similarly, $\mathbf{P}(BC) = 1 - p^2$. The event D to be able to drive to the city, I need to be able to drive from A to B and to drive from B to C, so $D = AB \cap BC$. By independence

$$\mathbf{P}(D) = \mathbf{P}(AB \cap BC) = \mathbf{P}(AB) \mathbf{P}(BC) = (1 - p^2)^2.$$

(b.) The event T that I can travel to the city is that I can drive or I can take the railway, so $T = S_5^c \cup D$. Using the union formula and independence,

$$\begin{aligned} \mathbf{P}(T) &= \mathbf{P}(S_5^c \cup D) = \mathbf{P}(S_5^c) + \mathbf{P}(D) - \mathbf{P}(S_5^c \cap D) \\ &= (1 - p) + (1 - p^2)^2 - (1 - p)(1 - p^2)^2 = 1 - p + p(1 - p^2)^2. \end{aligned}$$

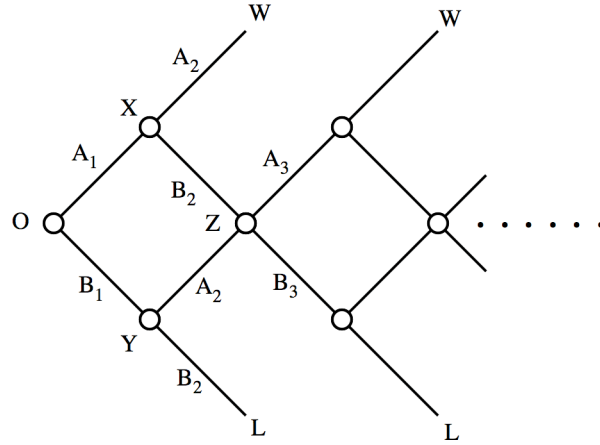
(c.) Using (a.) and (b.), independence and $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$,

$$\mathbf{P}(S_5|T) = \frac{\mathbf{P}(S_5 \cap T)}{\mathbf{P}(T)} = \frac{\mathbf{P}(S_5 \cap D)}{\mathbf{P}(T)} = \frac{\mathbf{P}(S_5) \mathbf{P}(D)}{\mathbf{P}(T)} = \frac{p(1 - p^2)^2}{1 - p + p(1 - p^2)^2}.$$

79[19] A and B play a sequence of independent games, in each of which A has the probability of p of winning and B has the probability of $q = 1 - p$ of winning.

- (a) Suppose that the sequence is won by the first player to achieve a lead of two games. Find the probability that A wins the tournament.
 (b) If the rules are changed so that the sequence is won by the first player to win two consecutive games, find the probability that A wins the tournament now.
 (c) Which of the rules give the weaker player a better chance of winning the tournament?

(a.) Let W denote the event that A wins the tournament and $L = W^c$ the event that B wins the tournament. The following diagram indicates the winner after a sequence of games. The branches of the tree are superimposed. The game starts at O . Let A_i denote



(a.) Player who leads by two games wins.

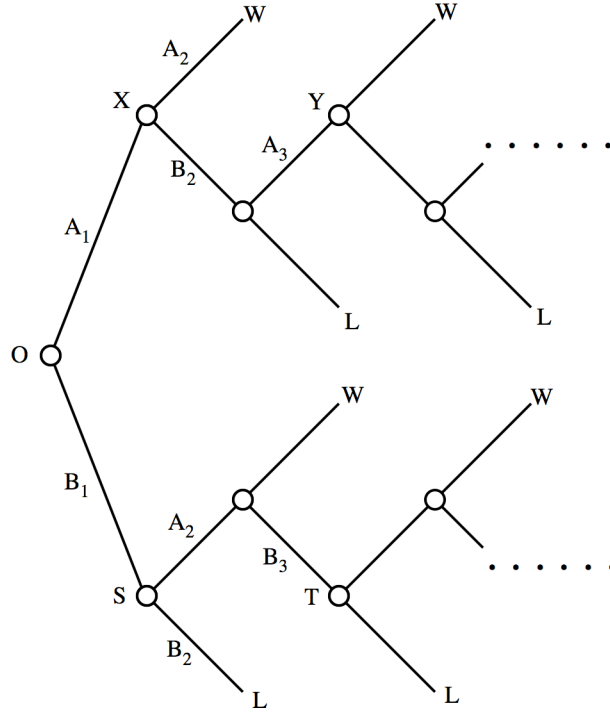
the event that A wins the i th game, and $B_i = A_i^c$ the event that B wins the i th game. The diagram is an infinite herring bone, thus looks exactly the same from the starting point O and when starting from point Z . Therefore $\mathbf{P}(W) = \mathbf{P}(W|Z)$. Z is reached given that A wins game one and loses game two, or loses game one and wins game two. On the other hand, using the first two games to partition W and independence,

$$\begin{aligned} \mathbf{P}(W) &= \mathbf{P}(W|A_1 \cap B_2) = \mathbf{P}(W|B_1 \cap A_2) \\ \mathbf{P}(W) &= \mathbf{P}(W \cap A_1 \cap A_2) + \mathbf{P}(W \cap A_1 \cap B_2) + \mathbf{P}(W \cap B_1 \cap A_2) + \mathbf{P}(W \cap B_1 \cap B_2) \\ &= \mathbf{P}(A_1 \cap A_2) \mathbf{P}(W|A_1 \cap A_2) + \mathbf{P}(A_1 \cap B_2) \mathbf{P}(W|A_1 \cap B_2) \\ &\quad + \mathbf{P}(B_1 \cap A_2) \mathbf{P}(W|B_1 \cap A_2) + \mathbf{P}(B_1 \cap B_2) \mathbf{P}(W|B_1 \cap B_2) \\ &= p^2 \cdot 1 + 2pq \mathbf{P}(W) + 0 \end{aligned}$$

since $\mathbf{P}(W|A_1 \cap A_2) = 1$ and $\mathbf{P}(W|B_1 \cap B_2) = 0$. Solving for $\mathbf{P}(W)$ gives

$$\mathbf{P}(W) = \frac{p^2}{1 - 2pq}.$$

(b.) The following diagram indicates the winner after a sequence of games by the new rules. This time we see the whole tree.



(b.) Player who takes two consecutive games wins.

This diagram has two infinite zig zags. It looks exactly the same given we're at the point X and when starting from point Y. Therefore $\mathbf{P}(W|A_1) = \mathbf{P}(W|A_1 \cap B_2 \cap A_3)$. On the other hand, using the second and third games to partition $W \cap A_1$,

$$\begin{aligned} \mathbf{P}(W|A_1) &= \mathbf{P}(W|A_1 \cap B_2 \cap A_3) \\ \mathbf{P}(W|A_1) &= \mathbf{P}(W \cap A_2|A_1) + \mathbf{P}(W \cap B_2 \cap A_3|A_1) + \mathbf{P}(W \cap B_2 \cap B_3|A_1) \\ &= \mathbf{P}(A_2) \mathbf{P}(W|A_2 \cap A_1) + \mathbf{P}(B_2 \cap A_3) \mathbf{P}(W|B_2 \cap A_3 \cap A_1) \\ &\quad + \mathbf{P}(B_2 \cap B_3) \mathbf{P}(W|B_2 \cap B_3 \cap A_1) \\ &= p + pq \mathbf{P}(W|A_1) + 0 \end{aligned}$$

since $\mathbf{P}(W|A_1 \cap A_2) = 1$ and $\mathbf{P}(W|B_2 \cap B_3 \cap A_1) = 0$. Solving for $\mathbf{P}(W|A_1)$ gives

$$\mathbf{P}(W|A_1) = \frac{p}{1 - pq}.$$

Similarly $\mathbf{P}(W|S) = \mathbf{P}(W|T)$. Since the lower zigzag is reflection of the upper, by swapping W and L swapping A_i with B_i and swapping p with q we have

$$\mathbf{P}(L|B_1) = \frac{q}{1 - pq}.$$

Finally, using the partitioning formula we find,

$$\begin{aligned} \mathbf{P}(W) &= \mathbf{P}(W \cap A_1) + \mathbf{P}(W \cap B_1) = \mathbf{P}(A_1) \mathbf{P}(W|A_1) + \mathbf{P}(B_1)(1 - \mathbf{P}(L|B_1)) \\ &= \frac{p^2}{1 - pq} + q \left(1 - \frac{q}{1 - pq}\right) = \frac{p^2(1 + q)}{1 - pq}. \end{aligned}$$

(c.) Suppose that A is the weaker player, $0 < p < \frac{1}{2}$. Because

$$p^2(1+q)(1-2pq) = p^2(1-2pq+q-2pq^2) = p^2(1-pq)+p^2q(1-p-2pq) = p^2(1-pq)+p^2q^2(1-2p)$$

and $1 - 2p > 0$ if $p < \frac{1}{2}$, we have then

$$p^2(1+q)(1-2pq) > p^2(1-pq)$$

so

$$\frac{p^2(1+q)}{1-pq} > \frac{p^2}{1-2pq}.$$

Thus A has a better chance of winning under the second rules.

79[19] *Irena throws at a target. After each throw she moves further away so that her probability of hitting is two thirds of the probability of a hit on the previous throw. The probability of a hit on the first throw is $\frac{1}{4}$. Find the probability of a hit on the n th throw. Deduce that the probability of never hitting the target is greater than $\frac{1}{4}$.*

Let H_i denote the event that she hits on the i th throw. We are given that $\mathbf{P}(H_1) = \frac{1}{4}$ and $\mathbf{P}(H_{i+1}) = \frac{2}{3} \mathbf{P}(H_i)$. It follows that

$$\mathbf{P}(H_i) = \frac{1}{4} \left(\frac{2}{3} \right)^{i-1}.$$

Using Boole's inequality and the sum of a geometric series, the probability of the event E of hitting the target on some shot is

$$\mathbf{P}(E) = \mathbf{P} \left(\bigcup_{i=1}^{\infty} H_i \right) \leq \sum_{i=1}^{\infty} \mathbf{P}(H_i) = \sum_{i=1}^{\infty} \frac{1}{4} \left(\frac{2}{3} \right)^{i-1} = \frac{\frac{1}{4}}{1 - \frac{2}{3}} = \frac{3}{4}.$$

In fact the events are not disjoint because we expect she has a positive probability of hitting both of any pair $i \neq j$, i.e., $\mathbf{P}(H_i \cap H_j) > 0$. Thus the inequality above is " $<$." It follows that never hitting has a probability greater than one quarter

$$\mathbf{P}(E^c) = 1 - \mathbf{P}(E) > 1 - \frac{3}{4} = \frac{1}{4}.$$

108[2] *A keen student has a algebra books, b books on boundary layers and c calculus books. If he places them on one shelf at random, what is the probability that:*

- (a) *Books on the same subject are not separated.*
- (b) *Books on the same subject are in alphabetical order, but not necessarily adjacent.*
- (c) *Books on the same subject are adjacent and in alphabetical order.*

(a.) The sample space Ω is the set of all permutations of the $a + b + c$ books, so $|\Omega| = (a + b + c)!$. If the subjects are not separated, they may be taken in $3!$ orders. Within the algebra books, there are $a!$ possible orderings, within the boundary layer books there are $b!$ orderings and within the calculus books, $c!$ orderings. Thus if D is the event that books are not separated,

$$\mathbf{P}(D) = \frac{|D|}{|\Omega|} = \frac{3! \cdot a! \cdot b! \cdot c!}{(a + b + c)!}.$$

(b.) Once we have identified where on the shelf the different subject books are placed, there is only one alphabetical ordering within the subject. Thus if E is the event that books in the same subject are alphabetically ordered, although not necessarily adjacent

$$\mathbf{P}(E) = \frac{|E|}{|\Omega|} = \frac{\binom{a+b+c}{a, b, c}}{(a+b+c)!} = \frac{(a+b+c)!}{a! \cdot b! \cdot c! \cdot (a+b+c)!} = \frac{1}{a! \cdot b! \cdot c!}.$$

(c.) There are $3!$ ways to order the subjects. Books within the subject can be in alphabetical order only in one way. If F is the event books are adjacent and in alphabetical order,

$$\mathbf{P}(F) = \frac{|F|}{|\Omega|} = \frac{3!}{(a+b+c)!}.$$