151[14] *Suppose X is a random variable that is uniform on $1 \le x \le m$. What is $\mathbf{P}(X = k \mid a \le X \le b)$? In particular, find $\mathbf{P}(X > n + k \mid X > n)$.*

The random variable X takes values in the set $D = \{1, 2, 3, \ldots, m\}$. The pmf of the uniform random variable is $f_X(x) = 1/m$ for $x \in D$ and $f_X(x) = 0$ if $x \notin D$. The answer is simpler if we make the assumption that a, k, b are integers and $1 \le a \le k \le b \le m$. Under this condition, all the numbers between a and b, inclusive, are in D so that there are $b - a + 1$ such numbers and $\mathbf{P}(a \le X \le b) = (b - a + 1)/m$. Similarly $a \le k \le b$ implies $\mathbf{P}(\{X = k\} \cap \{a \le X \le b\}) = \mathbf{P}(X = k) = 1/m$. Thus

$$\mathbf{P}(X = k \mid a \le X \le b) = \frac{\mathbf{P}(\{X = k\} \cap \{a \le X \le b\})}{\mathbf{P}(a \le X \le b)} = \frac{1}{b - a + 1}.$$

Assuming the conditions $0 \le n < m$ and $n \le n+k \le m$, the set of numbers in D that satisfy $X > n$ is $\{n + 1 \le X \le m\}$ so there are $m - n$ such numbers and $\mathbf{P}(X > n) = (m - n)/m$. Also the set of numbers in D that satisfy $\{X > n + k\} \cap \{X > n\} = \{X > n + k\}$ is $\{n + k + 1 \le X \le m\}$ so there are $m - n - k$ such numbers and $\mathbf{P}(\{X > n + k\} \cap \{X > n\}) = (m - n - k)/m$. Thus

$$\mathbf{P}(X > n + k \mid X > n) = \frac{\mathbf{P}(\{X > n + k\} \cap \{X > n\})}{\mathbf{P}(X > n)} = \frac{m - n - k}{m - n}.$$

We also give a solution in case we make weaker hypotheses on the numbers a, b, k which are not specified in this problem. Since we are conditioning on the event $\{a \le X \le b\}$, which must have positive probability, at least one of the numbers from D have to be included between a and b. In other words, we can assume the weaker conditions $a \le b$, $a \le m$ and $b \ge 1$. Thus the numbers in D that are between a and b are exactly

$$\max(a, 1), \max(a, 1) + 1, \ldots, \min(b, m).$$

For example if $m = 6$ as in X is the number on one roll of a die, and $a = 2$, $b = 9$, then the possible values of X between a and b are $\max(2, 1) = 2, 3, 4, 5, 6 = \min(9, 6)$. The probability is thus the number of numbers times the probability of any one of them, or

$$\mathbf{P}(a \le X \le b) = \frac{\min(b, m) - \max(a, 1) + 1}{m}. \tag{1}$$

For example on the standard die with $a = 2$, $b = 9$ this is $(6 - 2 + 1)/6$. The intersection event $\{X = k\} \cap \{a \le X \le b\} = \{X = k\}$ if $\max(a, 1) \le k \le \min(b, m)$ and the empty set if not. Thus the conditional probability

$$\begin{aligned}\mathbf{P}(X = k \mid a \le X \le b) &= \frac{\mathbf{P}(\{X = k\} \cap \{a \le X \le b\})}{\mathbf{P}(a \le X \le b)} \\ &= \begin{cases} \dfrac{\mathbf{P}(X = k)}{\mathbf{P}(a \le X \le b)}, & \text{if } \max(a, 1) \le k \le \min(b, m); \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \dfrac{1}{\min(b, m) - \max(a, 1) + 1}, & \text{if } \max(a, 1) \le k \le \min(b, m); \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

For the second problem, we assume $n < m$ so that $\{X > n\} \cap D \neq \emptyset$ so it has positive probability. Using (1),

$$\begin{aligned}
\mathbf{P}(X > n+k \mid X > n) &= \frac{\mathbf{P}(\{X > n+k\} \cap \{X > n\})}{\mathbf{P}(X > n)} = \frac{\mathbf{P}(X > n+k)}{\mathbf{P}(X > n)} \\
&= \begin{cases} \dfrac{\mathbf{P}(n+k < X \le m)}{\mathbf{P}(n < X \le m)}, & \text{if } m > n+k; \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \dfrac{m - \max(n+k+1,1)+1}{m-\max(n+1,1)+1}, & \text{if } m > n+k; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

151[42] *Prove Chebychev's Inequality, that for a random variable X with mean μ and variance σ^2,*

$$\mathbf{P}(|X-\mu| \le h\sigma) \ge 1 - \frac{1}{h^2} \qquad \textit{for any } h > 0. \tag{2}$$

When an unbiased coin is tossed n times, let the number of heads be m. Show that

$$\mathbf{P}\left(0.4 \le \frac{m}{n} \le 0.6\right) \ge 0.75$$

when $n \ge 100$. Given that $n = 100$, show that the actual probability is

$$\mathbf{P}\left(0.49 \le \frac{m}{n} \le 0.51\right) \simeq \frac{3}{5\sqrt{2\pi}}.$$

You may assume Stirling's Formula $n! \simeq \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}$.

Applying Theorem 4.6.1 to $h(x) = (x - \mathbf{E}(X))^2$, we get the version of Chebychev's inequality given in class,

$$\mathbf{P}(|X - \mathbf{E}(X)| \ge a) \le \frac{\mathbf{Var}(X)}{a^2}, \qquad \text{for any } a > 0. \tag{3}$$

The desired inequality reverses signs, so we expect to apply it to the complementary event. Furthermore, we replace $a = h\sigma$, use the fact that the event $\{|X-\mu| > h\sigma\} \subset \{|X-\mu| \ge h\sigma\}$ and (3),

$$\begin{aligned}
1 - \mathbf{P}(|X-\mu| \le h\sigma) &= \mathbf{P}(\{|X-\mu| \le h\sigma\}^c) \\
&= \mathbf{P}(|X-\mu| > h\sigma) \\
&\le \mathbf{P}(|X-\mu| \ge h\sigma) \\
&\le \frac{\sigma^2}{(h\sigma)^2} = \frac{1}{h^2}.
\end{aligned}$$

Rearranging gives (2).

The second question asks us to apply the inequality to the random variable m, the number of heads in n tosses, whch has the distribution of a binomial random variable $m \sim \text{binomial}(n, p = \frac{1}{2})$. From Table 4.1, $\mu = \mathbf{E}(m) = np = 0.5n$ and $\sigma^2 = \mathbf{Var}(m) = npq = 0.25n$ so $\sigma = 0.5\sqrt{n}$. We may rewrite the event using equivalent inequalities on m,

$$\left\{0.4 \le \frac{m}{n} \le 0.6\right\} = \{0.4n \le m \le 0.6n\} = \{-0.1n \le m - 0.5n \le 0.1n\} = \{|m-\mu| \le 0.1n\}.$$

Thus $0.1n = h\sigma = 0.5h\sqrt{n}$ so $h = 0.2\sqrt{n}$. Thus applying (2),

$$\mathbf{P}\left(0.4 \le \frac{m}{n} \le 0.6\right) = \mathbf{P}(|m-\mu| \le 0.1n = h\sigma) \ge 1 - \frac{1}{h^2} = 1 - \frac{1}{0.04n} \ge 1 - \frac{1}{4} = 0.75$$

if $n \ge 100$.

The third question asks to compute the exact probability in case $m \sim \text{bin}(n = 100, p = \frac{1}{2})$. The desired event

$$B = \left\{0.49 \le \frac{m}{n} \le 0.51\right\} = \{49 \le m \le 51\}.$$

The pmf for $x \in D = \{0, 1, 2, \ldots, 100\}$ is

$$f_m(x) = \binom{n}{x} p^x q^{n-x} = \binom{100}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{100-x} = \binom{100}{x} \frac{1}{2^{100}}.$$

Thus the probability is approximated using Stirling's Formula (and answer uses $151/51 \approx 3$).

$$\begin{aligned}
\mathbf{P}(B) &= f_m(49) + f_m(50) + f_m(51) \\
&= \frac{1}{2^{100}}\binom{100}{49} + \frac{1}{2^{100}}\binom{100}{50} + \frac{1}{2^{100}}\binom{100}{51} \\
&= \frac{1}{2^{100}}\left(\frac{100!}{49!\,51!} + \frac{100!}{50!\,50!} + \frac{100!}{51!\,49!}\right) \\
&= \frac{100!}{2^{100}\,(50!)^2}\left(\frac{50}{51} + 1 + \frac{50}{51}\right) \\
&\approx \frac{\sqrt{2\pi}\,100^{100.5}e^{-100}}{2^{100}\,(\sqrt{2\pi}\,50^{50.5}\,e^{-50})^2} \cdot \frac{151}{51} \\
&= \frac{10}{\sqrt{2\pi}\,50} \cdot \frac{151}{51} = \frac{151}{255\sqrt{2\pi}} \approx 0.236.
\end{aligned}$$

[A.] *Let X be a random variable with finite second moment. Let $\mu = \mathbf{E}(X)$ and $\sigma^2 = \mathbf{Var}(X)$. Let $Y = (X - \mu)/\sigma$. Find $\mathbf{E}(Y)$ and $\mathbf{Var}(Y)$.*

Finiteness of second moment implies the variance and expectation are defined. We are given $Y = aX + b$ where $a = 1/\sigma$ and $b = -\mu/\sigma$. Using the formula for a linear change in the expectation,

$$\mathbf{E}(Y) = \mathbf{E}(aX + b) = a\,\mathbf{E}(X) + b = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = 0.$$

Similarly, using the formula for a linear change of variable for the variance,

$$\mathbf{Var}(Y) = \mathbf{Var}(aX + b) = a^2\,\mathbf{Var}(X) = \left(\frac{1}{\sigma}\right)^2 \sigma^2 = 1.$$

[B.] *Roll five dice. Let X be the smallest of the numbers rolled. Find, $\mathbf{P}(X \ge x)$, $f_X(x)$ and $\mathbf{E}(X)$.*

The sample space $\Omega = \{(x_1, \ldots, x_5) \in \mathbb{N}^5 : \ x_i \le 6 \text{ for all } i\}$, five-tuples of numbers one to six. Each outcome of five rolls are equally likely and $|\Omega| = 6^5$. The set of possible values taken by the random variable X is $D = \{1, 2, 3, 4, 5, 6\}$. The key observation is that the event $E_k = \{X \ge k\}$, that the least number on all rolls is k, is equivalent to each of the dice showing a number at least k, or $E_k = \{(x_1, \ldots, x_5) \in \Omega : \ k \le x_i \le 6 \text{ for all } i\}$. It follows that $|E_k| = (7-k)^5$ since there are $6 - k + 1$ numbers between k and 6 inclusive. Thus for $k \in D$,

$$\mathbf{P}(X \ge k) = \frac{(7-k)^5}{6^5}.$$

To find the pmf, observe that the event $\{X = k\} = \{X \ge k\} \setminus \{X \ge k+1\}$. It follows that the pmf for $k \in D$ is

$$f_X(k) = \mathbf{P}(X = k) = \mathbf{P}(X \ge k) - \mathbf{P}(X \ge k+1) = \frac{(7-k)^5}{6^5} - \frac{(7-(k+1))^5}{6^5}$$

Using the alternative formula for expectation of a nonnegative integer valued random variable Theorem 4.3.11,

$$\mathbf{E}(X) = \sum_{k=1}^{6} \mathbf{P}(X \ge k) = \sum_{k=1}^{6} \frac{(7-k)^5}{6^5} = \frac{6^5 + 5^5 + 4^5 + 3 + 2^5 + 1^5}{6^5} = \frac{12201}{7776} \approx 1.5609.$$

[C.] *Suppose the random variable X is distributed according to the Poisson distribution with mean λ. Find $f_X(x \mid X \text{ is odd })$ and $\mathbf{E}(X \mid X \text{ is odd })$.*

The values of the Poisson variable are taken in $D = \{0, 1, 2, 3, \ldots\}$ and for $\lambda > 0$, the pmf for $x \in D$ is

$$f_X(x) = \mathbf{P}(X = x) = \frac{e^{-\lambda}\,\lambda^x}{x!}.$$

We need to consider the event

$$\{X \text{ is odd}\} = \{X = 1\} \cup \{X = 3\} \cup \{X = 5\} \cup \cdots = \bigcup_{k=0}^{\infty} \{X = 2k+1\}.$$

Its probability of being odd turns out to be less than one half

$$\mathbf{P}(X \text{ is odd}) = \sum_{k=0}^{\infty} f_X(2k+1) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\,\lambda^{2k+1}}{(2k+1)!} = e^{-\lambda} \sinh\lambda = e^{-\lambda}\left(\frac{e^{\lambda} - e^{-\lambda}}{2}\right) = \frac{1 - e^{-2\lambda}}{2}$$

To compute the conditional mass function, we use the usual formula for conditional probability. If $x \in D$,

$$\begin{aligned} f_X(x \mid X \text{ is odd}) = \mathbf{P}(X = x \mid X \text{ is odd}) &= \frac{\mathbf{P}(\{X = x\} \cap \{X \text{ is odd}\})}{\mathbf{P}(X \text{ is odd})} \\ &= \begin{cases} \dfrac{\mathbf{P}(X = x)}{\mathbf{P}(X \text{ is odd})}, & \text{if } x \text{ is odd;} \\ 0, & \text{if } x \text{ is even.} \end{cases} \\ &= \begin{cases} \dfrac{\lambda^x}{x!\,\sinh\lambda}, & \text{if } x \text{ is odd;} \\ \\ 0, & \text{if } x \text{ is even.} \end{cases} \end{aligned}$$

The conditional expectation is given by the formula

$$\begin{aligned} \mathbf{E}(X \mid X \text{ is odd}) &= \sum_{x \in D} x f_X(x \mid X \text{ is odd}) \\ &= \sum_{k=0}^{\infty} \frac{(2k+1)\,\lambda^{2k+1}}{(2k+1)!\,\sinh\lambda} = \frac{\lambda}{\sinh\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = \frac{\lambda\,\cosh\lambda}{\sinh\lambda} = \lambda \coth\lambda. \end{aligned}$$