226[9] *Let $X_n \in \{1, -1\}$ be a sequence of independent random variables such $\mathbf{P}(X_n = 1) = p = 1 - q = 1 - \mathbf{P}(X_n = -1)$. Let U be the number of terms in the sequence before the first change of sign and V the further number of terms before the second change of sign. In other words, the sequence $X_1, X_2, \ldots$ of random variables is made up of a number of runs of $+1$'s and runs of -1's. U is the length of the first run and V is the length of the second run.*

(a) *Show that $\mathbf{E}(U) = \dfrac{p}{q} + \dfrac{q}{p}$ and $\mathbf{E}(V) = 2$.*

(b) *Write down the joint distribution of U and V and find $\mathbf{Cov}(U, V)$ and $\rho(U, V)$.*

(a.) To compute the expectations, let us determine the pmf's for the individual variables U and V following the procedure outlined in lecture. Observe that set of values of U or V, the number in the run is $D_i = \mathbb{N} = \{1, 2, 3, \ldots\}$, the natural numbers. Let A_i denote the event that $X_i = 1$. The idea is to condition on A_1, the first value. This determines whether the first run is $+1$'s or -1's. Thus if $X_1 = 1$ and there are u in the first run, then $X_2 = X_3 = \cdots = X_u = 1$ and $X_{u+1} = -1$, so if $u \in D_1$,

$$f_U(u \mid A_1) = p^{u-1} q.$$

Similarly, if $X_1 = -1$, then

$$f_U(u \mid A_1^c) = q^{u-1}\, p.$$

Using the partitioning formula

$$f_U(u) = f_U(u \mid A_1)\,\mathbf{P}(A_1) + f_U(u \mid A_1^c)\,\mathbf{P}(A_1^c) = p^{u-1}\, q\, p + p\, q^{u-1}\, q = p^u\, q + p\, q^u.$$

It follows that

$$\mathbf{P}(U \ge u) = \sum_{k=u}^{\infty} \left(p^k\, q + p\, q^k\right) = \frac{p^u q}{1-p} + \frac{p\, q^u}{1-q} = p^u + q^u.$$

Thus, using Theorem 4.3.11,

$$\mathbf{E}(U) = \sum_{u=1}^{\infty} \mathbf{P}(U \ge u) = \sum_{u=1}^{\infty} (p^u + q^u) = \frac{p}{1-p} + \frac{q}{1-q}.$$

Thus if $X_1 = 1$ and there are u in the first run and v in the second run, then $X_{u+1} = -1$ and $X_{u+2} = X_{u+3} = \cdots = X_{u+v} = -1$ and $X_{u+v+1} = 1$, so if $v \in D_2$, and independence of individual X_i's,

$$f_V(v \mid A_1) = p\, q^{v-1}.$$

Similarly, if $X_1 = -1$, then

$$f_V(v \mid A_1^c) = p^{v-1}\, q.$$

Using the partitioning formula

$$f_V(v) = f_V(v \mid A_1)\,\mathbf{P}(A_1) + f_V(v \mid A_1^c)\,\mathbf{P}(A_1^c) = p\, q^{v-1}\, p + p^{v-1}\, q^2 = p^2\, q^{v-1} + p^{v-1}\, q^2.$$

Here is an alternative way to compute the expectation. These formulas involve for $|z| < 1$,

$$\sum_{k=1}^{\infty} k z^{k-1} = \frac{d}{dz}\left(\sum_{k=0}^{\infty} z^k\right) = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2};$$

$$\sum_{k=1}^{\infty} k^2 z^{k-1} = \sum_{k=1}^{\infty}[(k+1)k - k]z^{k-1} = \frac{d^2}{dz^2}\left(\sum_{k=-1}^{\infty} z^{k+1}\right) - \frac{1}{(1-z)^2} \qquad (1)$$

$$= \frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) - \frac{1}{(1-z)^2} = \frac{2}{(1-z)^3} - \frac{1}{(1-z)^2} = \frac{1+z}{(1-z)^3}.$$

Thus, using (1),

$$\mathbf{E}(V) = \sum_{v\in D_2} v\, f_V(v) = \sum_{v=1}^{\infty} v\left(p^2\, q^{v-1} + p^{v-1}\, q^2\right)$$

$$= p^2 \sum_{v=0}^{\infty} v q^{v-1} + q^2 \sum_{v=0}^{\infty} v p^{v-1} = \frac{p^2}{(1-q)^2} + \frac{q^2}{(1-p)^2} = 2.$$

(b.) The joint probability is also gotten by conditioning on A_1. Let $(u,v) \in D_1 \times D_2$ and $f(u,v) = \mathbf{P}(U = u \text{ and } V = v)$. Thus if $X_1 = 1$ and the length of the first run is u and the length of the second run is v, then $X_2 = \cdots = X_u = 1$, $X_{u+1} = X_{u+2} = \cdots = X_{u+v} = -1$ and $X_{u+v+1} = 1$, so if $v \in D_2$, and independence of individual X_i's,

$$f(u, v \mid A_1) = p^{u-1} q^v p = p^u q^v.$$

Similarly, if $X_1 = -1$, then

$$f(u, v \mid A_1^c) = q^{u-1}\, p^v\, q = p^v\, q^u.$$

Using the partitioning formula

$$f(u,v) = f_V(u,v|A_1)\,\mathbf{P}(A_1) + f(u,v|A_1^c)\,\mathbf{P}(A_1^c) = p^u\, q^v\, p + p^v\, q^u\, q = p^{u+1} q^v + p^v q^{u+1}.$$

Let us check the marginal probabilities.

$$f_U(u) = \sum_{v\in D_2} f(u,v) = \sum_{v=1}^{\infty}\left(p^{u+1}q^v + p^v q^{u+1}\right) = \frac{p^{u+1}q}{1-q} + \frac{pq^{u+1}}{1-p} = p^u q + pq^u;$$

$$f_V(v) = \sum_{u\in D_1} f(u,v) = \sum_{u=1}^{\infty}\left(p^{u+1}q^v + p^v q^{u+1}\right) = \frac{p^2\, q^v}{1-p} + \frac{p^v\, q^2}{1-q} = p^2\, q^{v-1} + p^{v-1}\, q^2.$$

To compute the further expectations, using (1),

$$\mathbf{E}(U^2) = \sum_{u=1}^{\infty} u^2\, f_U(u) = \sum_{u=1}^{\infty} u^2 (p^u q + pq^u) = \frac{pq(1+p)}{(1-p)^3} + \frac{pq(1+q)}{(1-q)^3} = \frac{p(1+p)}{q^2} + \frac{q(1+q)}{p^2}$$

$$\mathbf{E}(V^2) = \sum_{v=1}^{\infty} v^2\, f_V(v) = \sum_{v=1}^{\infty} v^2 (p^2 q^{v-1} + p^{v-1} q^2) = \frac{p^2(1+q)}{(1-q)^3} + \frac{q^2(1+p)}{(1-p)^3} = \frac{1+q}{p} + \frac{1+p}{q}$$

$$\mathbf{E}(UV) = \sum_{u=1}^{\infty}\sum_{v=1}^{\infty} uv\, f(u,v) = \sum_{u=1}^{\infty}\sum_{v=1}^{\infty} uv(p^{u+1}q^v + p^v q^{u+1}) = \frac{p^2 q + pq^2}{(1-p)^2(1-q)^2} = \frac{1}{q} + \frac{1}{p}.$$

The variances are

$$\mathbf{Var}(U) = \mathbf{E}(U^2) - \mathbf{E}(U)^2 = \frac{p(1+p)}{q^2} + \frac{q(1+q)}{p^2} - \left(\frac{p}{q} + \frac{q}{p}\right)^2;$$
$$\mathbf{Var}(V) = \mathbf{E}(V^2) - \mathbf{E}(V)^2 = \frac{1+q}{p} + \frac{1+p}{q} - 4;$$
$$\mathbf{Cov}(U,V) = E(UV) - \mathbf{E}(U)\,\mathbf{E}(V) = \frac{1}{q} + \frac{1}{p} - \frac{2p}{q} - \frac{2q}{p} = \frac{p+q-2p^2-2q^2}{pq}.$$

The expressions may be simplified using

$$p^2 + q^2 = p^2 + 2pq + q^2 - 2pq = (p+q)^2 - 2pq = 1 - 2pq; \quad (2)$$
$$p^2 - q^2 = (p+q)(p-q) = p - q. \quad (3)$$

The variance of U may be simplified using (2) and (3),

$$\begin{aligned}\mathbf{Var}(U) &= \frac{p^3(p+q+p) + q^3(p+q+q)}{p^2q^2} - \left(\frac{p^2+q^2}{pq}\right)^2\\ &= \frac{2p^4 + p^3q + pq^3 + 2q^4 - p^4 - 2p^2q^2 - q^4}{p^2q^2}\\ &= \frac{(p^4 - 2p^2q^2 + q^4) + pq(p^2+q^2)}{p^2q^2}\\ &= \frac{(p^2-q^2)^2 + pq(p^2 - 2pq + q^2) + 2p^2q^2}{p^2q^2}\\ &= 2 + \frac{(p-q)^2}{pq} + \frac{(p-q)^2}{p^2q^2}.\end{aligned}$$

The variance of V may be simplified

$$\begin{aligned}\mathbf{Var}(V) &= \frac{p+2q}{p} + \frac{2p+q}{q} - 4 = \frac{pq + 2q^2 + 2p^2 + pq - 4pq}{pq}\\ &= \frac{2pq + 2(p^2 - 2pq + q^2)}{pq} = 2 + \frac{2(p-q)^2}{pq}.\end{aligned}$$

Thus $\mathbf{Var}(V) < \mathbf{Var}(U)$ unless $p = q = \frac{1}{2}$.
The covariance may be simplified using (2)

$$\begin{aligned}\mathbf{Cov}(U,V) &= \frac{p+q-2p^2-2q^2}{pq} = \frac{1-2(p^2+q^2)}{pq} = \frac{1-2(1-2pq)}{pq} = \frac{4pq-1}{pq}\\ &= \frac{4pq - (p+q)^2}{pq} = -\frac{(p-q)^2}{pq}.\end{aligned}$$

The correlation coefficient is thus

$$\begin{aligned}\rho(U,V) &= \frac{\mathbf{Cov}(U,V)}{\sqrt{\mathbf{Var}(U)} \cdot \sqrt{\mathbf{Var}(V)}}\\ &= \frac{-(p-q)^2}{\sqrt{\left(2pq + (p-q)^2 + \frac{(p-q)^2}{pq}\right)(2pq + 2(p-q)^2)}}.\end{aligned}$$

I doubt that the text's answer is correct since neither variance has $(p-q)^2$ as a factor.

226[25] *Let X and Y be independent geometric variables so that for $m \ge 0$,*

$$f_X(m) = \mathbf{P}(X = m) = (1-\lambda)\lambda^m, \qquad f_Y(m) = \mathbf{P}(Y = m) = (1-\mu)\mu^m,$$

where $0 < \lambda, \mu < 1$.

(a) If $\lambda \ne \mu$, show that

$$\mathbf{P}(X + Y = n) = \frac{(1-\lambda)(1-\mu)}{\lambda - \mu}\left(\lambda^{n+1} - \mu^{n+1}\right).$$

Find $\mathbf{P}(X = k \mid X + Y = n)$.

(b) Find the distribution of $Z = X + Y$ if $\lambda = \mu$, and show that in this case, $\mathbf{P}(X = k \mid X + Y = n) \; = \dfrac{1}{n+1}$.

These geometric rv's are defined for $m \in D = \{0, 1, 2, \ldots\}$. Since the variables are assumed independent, their joint pmf is the product

$$f(x, y) = f_X(x)\, f_Y(y) = (1-\lambda)\,(1-\mu)\,\lambda^x\,\mu^y.$$

(a.) The pmf of the sum of independent variables is given by the convolution formula Theorem 5.4.11. We observe that the sum is a finite geometric sum.

$$\begin{aligned}
f_Z(z) &= \sum_{x=0}^{z} f_X(x)\, f_Y(z-x) \\
&= \sum_{x=0}^{z} (1-\lambda)\,(1-\mu)\,\lambda^x\,\mu^{z-x} \\
&= (1-\lambda)\,(1-\mu)\,\mu^z \sum_{x=0}^{z} \left(\frac{\lambda}{\mu}\right)^x \\
&= (1-\lambda)\,(1-\mu)\,\mu^z\, \frac{1 - \left(\frac{\lambda}{\mu}\right)^{z+1}}{1 - \frac{\lambda}{\mu}} \\
&= (1-\lambda)\,(1-\mu)\,\frac{\mu^{z+1} - \lambda^{z+1}}{\mu - \lambda}.
\end{aligned}$$

Observing that $x = k$ and $x + y = n$ implies $x = k$ and $y = n - k$, the conditional probability is gotten using the usual formula for $0 \le k \le n$,

$$\begin{aligned}
\mathbf{P}(X = k \mid Z = n) &= \frac{\mathbf{P}(X = k \text{ and } Z = n)}{\mathbf{P}(Z = n)} = \frac{\mathbf{P}(X = k \text{ and } Y = n - k)}{\mathbf{P}(Z = n)} \\
&= \frac{f(k, n-k)}{f_Z(n)} = \frac{\lambda^k\,\mu^{n-k}(\lambda - \mu)}{\lambda^{n+1} - \mu^{n+1}}.
\end{aligned}$$

(b.) In case $\lambda = \mu$,

$$\begin{aligned}
f_Z(z) &= \sum_{x=0}^{z} f_X(x)\, f_Y(z-x) \\
&= \sum_{x=0}^{z} (1-\lambda)^2\,\lambda^x\,\lambda^{z-x} \\
&= (z+1)\,(1-\lambda)^2\,\lambda^z.
\end{aligned}$$

The conditional probability is for $0 \le k \le n$,

$$\mathbf{P}(X = k \mid Z = n) = \frac{\mathbf{P}(X = k \text{ and } Z = n)}{\mathbf{P}(Z = n)} = \frac{\mathbf{P}(X = k \text{ and } Y = n - k)}{\mathbf{P}(Z = n)}$$
$$= \frac{f(k, n-k)}{f_Z(n)} = \frac{(1-\lambda)^2 \lambda^n}{(n+1)(1-\lambda)^2 \lambda^n} = \frac{1}{n+1}.$$

[A.] *Two cards are chosen at random without replacement from a standard deck. Let X denote the number of kings and Y the number of clubs. Find the joint pmf $f(x, y)$, $\mathbf{Cov}(X, Y)$ and $\rho(X, Y)$.*

The sample space Ω is the set of combinations of 52 cards taken two at a time. Thus $|\Omega| = \binom{52}{2} = 1326$. Both X and Y take values in $D_i = \{0, 1, 2\}$. The pairs take values in $D_1 \times D_2$. If $X = x$ and $Y = y$ then $f(x, y) = \mathbf{P}(X = x \text{ and } Y = y)$. There are $52 - 16 = 36$ cards that are neither kings nor clubs. If $X = 0$ and $Y = 0$ both cards are neither king nor club. If instead $Y = 1$ then one card is a club that isn't a king and the other is neither king nor club. If also $Y = 2$ both cards are clubs but neither is a king.

$$f(0,0) = \frac{\binom{36}{2}}{\binom{52}{2}} = \frac{36 \cdot 35}{52 \cdot 51} = \frac{105}{221}, \qquad f(0,1) = \frac{12 \cdot 36}{\binom{52}{2}} = \frac{12 \cdot 36 \cdot 2}{52 \cdot 51} = \frac{72}{221}$$
$$f(0,2) = \frac{\binom{12}{2}}{\binom{52}{2}} = \frac{12 \cdot 11}{52 \cdot 51} = \frac{11}{221}$$

If $X = 1$ there is one king. For $Y = 0$ the king can't be a club so there are three remaining kings. The second card cannot be king nor club, so there are 36 choices. For $Y = 1$ there is one king and one club. Either one card is the king of clubs and the other neither king nor club or one is a non-club king and the other is a non-king club. If also $Y = 2$ then one of the cards is a king of clubs and the other is another club so

$$f(1,0) = \frac{3 \cdot 36}{\binom{52}{2}} = \frac{3 \cdot 36 \cdot 2}{52 \cdot 51} = \frac{18}{221}, \qquad f(1,1) = \frac{1 \cdot 36 + 3 \cdot 12}{\binom{52}{2}} = \frac{72 \cdot 2}{52 \cdot 51} = \frac{12}{221},$$
$$f(1,2) = \frac{1 \cdot 12}{\binom{52}{2}} = \frac{12 \cdot 2}{52 \cdot 51} = \frac{2}{221}$$

If $X = 2$ and $Y = 0$ both cards are kings but neither is the king of clubs. If instead $Y = 1$ then there are two kings, one being the king of clubs. If also $Y = 2$ then it is impossible that both cards are kings and both cards are clubs.

$$f(2,0) = \frac{\binom{3}{2}}{\binom{52}{2}} = \frac{3 \cdot 2}{52 \cdot 51} = \frac{1}{442}, \qquad f(2,1) = \frac{3}{\binom{52}{2}} = \frac{3 \cdot 2}{52 \cdot 51} = \frac{1}{442}, \qquad f(2,2) = 0.$$

The joint pmf is collected in Figure 1.

The marginal probabilities are the row and column sums of the joint pmf. For $x \in D_1$ or $y \in D_2$,

$$f_X(x) = \sum_{y \in D_2} f(x, y); \qquad f_Y(y) = \sum_{x \in D_1} f(x, y).$$

The marginal probabilities are also given in Figure 1.

The variables X and Y are not independent, for example because

$$f(2,2) = 0 \ne \frac{1}{221} \cdot \frac{13}{221} = f_X(2) f_Y(2).$$

	$x=0$	$x=1$	$x=2$	$f_Y(y)$
$y=0$	$\frac{105}{221}$	$\frac{18}{221}$	$\frac{1}{442}$	$\frac{247}{442}=\frac{19}{34}$
$y=1$	$\frac{72}{221}$	$\frac{12}{221}$	$\frac{1}{442}$	$\frac{169}{442}=\frac{13}{34}$
$y=2$	$\frac{11}{221}$	$\frac{2}{221}$	0	$\frac{13}{221}=\frac{1}{17}$
$f_X(x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$	1

Figure 1: Table of joint pmf and marginal probabilities.

The expected values are

$$\begin{aligned}
\mathbf{E}(X) &= \sum_{x\in D_1} x\, f_X(x) = \frac{0\cdot 188 + 1\cdot 32 + 2\cdot 1}{221} = \frac{34}{221} = \frac{2}{13};\\
\mathbf{E}(Y) &= \sum_{y\in D_2} y\, f_Y(y) = \frac{0\cdot 19 + 1\cdot 13 + 2\cdot 2}{34} = \frac{17}{34} = \frac{1}{2};\\
\mathbf{E}(X^2) &= \sum_{x\in D_1} x^2\, f_X(x) = \frac{0^2\cdot 188 + 1^2\cdot 32 + 2^2\cdot 1}{221} = \frac{36}{221};\\
\mathbf{E}(Y^2) &= \sum_{y\in D_2} y^2\, f_Y(y) = \frac{0^2\cdot 19 + 1^2\cdot 13 + 2^2\cdot 2}{34} = \frac{21}{34};\\
\mathbf{E}(XY) &= \sum_{(x,y)\in D_1\times D_2} x\, y\, f(x,y) = \frac{1\cdot 1\cdot 24 + 1\cdot 2\cdot 4 + 2\cdot 1\cdot 1}{442} = \frac{34}{442} = \frac{1}{13}.
\end{aligned}$$

The variances and covariances are from their computational formulas,

$$\begin{aligned}
\mathbf{Var}(X) &= \mathbf{E}(X^2) - \mathbf{E}(X)^2 = \frac{36}{221} - \frac{2^2}{13^2} = \frac{400}{2873};\\
\mathbf{Var}(Y) &= \mathbf{E}(Y^2) - \mathbf{E}(Y)^2 = \frac{21}{34} - \frac{1^2}{2^2} = \frac{25}{68};\\
\mathbf{Cov}(X,Y) &= \mathbf{E}(XY) - \mathbf{E}(X)\,\mathbf{E}(Y) = \frac{1}{13} - \frac{2}{13}\cdot\frac{1}{2} = 0.
\end{aligned}$$

Thus X and Y are uncorrelated, since the correlation coefficient is

$$\rho(X,Y) = \frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)}\,\sqrt{\mathbf{Var}(Y)}} = 0.$$