

(1.) A standard deck of 52 cards consists of four suits $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Each suit has 13 different kinds of cards $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. The court cards J, Q and K count 10, the ace A counts 11. Suppose that you deal two cards from the deck without replacement. What is the probability that their sum is 21? Suppose that you deal four cards without replacement, two cards to your opponent two cards to yourself. Your opponent reveals that the sum of his cards is 21. What is the probability that your sum is 21 now? What is the probability that you both have 21? Why?

The sample space Ω consists of all two card subsets so $|\Omega| = \binom{52}{2}$ (order is not important). The event B that the cards total 21 can occur exactly when there is one ace and one of the court cards 10, J, Q or K. Since there are four suits, there are 4 ways to choose an ace and 16 ways to choose a court card, so $|B| = 4 \times 16$ so

$$\mathbf{P}(B) = \frac{4 \cdot 16}{\binom{52}{2}} = \frac{4 \cdot 16 \cdot 2}{52 \cdot 51} = \frac{32}{13 \cdot 51} \approx 0.04827.$$

Let B denote the event that your opponent has 21 and A the event that you have 21. Given that your opponent has 21, there is one less ace and one less court card in the deck for your hand, to be chosen from the remaining 50 cards, so

$$\mathbf{P}(A|B) = \frac{3 \cdot 15}{\binom{50}{2}} = \frac{3 \cdot 15 \cdot 2}{50 \cdot 49} = \frac{9}{5 \cdot 49} \approx 0.03673.$$

Using the product formula

$$\mathbf{P}(A \cap B) = \mathbf{P}(B) \mathbf{P}(A|B) = \frac{32}{\binom{52}{2}} \frac{45}{\binom{50}{2}} = \frac{32 \cdot 45 \cdot 2 \cdot 2}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{9}{5 \cdot 49} \approx 0.001773.$$

(2.) Certain circuit boards produced by Elberta Electric are rated excellent, acceptable, or unacceptable. Suppose that 30% of these are excellent, 60% are acceptable and 10% are unacceptable. Further, suppose that 10% of the excellent boards fail, 20% of the acceptable boards fail and 100% of the unacceptable boards fail (unacceptable boards are discarded without being used). Given that the board fails, what is the probability that it was rated excellent?

Let Ω denote these circuit boards. Let E , A and U be the excellent, acceptable and unacceptable boards, resp. Thus E , A and U are disjoint events that exhaust Ω . We are given $\mathbf{P}(E) = 0.3$, $\mathbf{P}(A) = 0.6$ and $\mathbf{P}(U) = 0.1$. Let F denote the event that the circuit board fails. We are given $\mathbf{P}(F|E) = 0.1$, $\mathbf{P}(F|A) = 0.2$ and $\mathbf{P}(F|U) = 1.0$. Then, Bayes' Theorem says

$$\begin{aligned} \mathbf{P}(E|F) &= \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}((E \cap F) \cup (A \cap F) \cup (U \cap F))} \\ &= \frac{\mathbf{P}(E) \mathbf{P}(F|E)}{\mathbf{P}(E) \mathbf{P}(F|E) + \mathbf{P}(A) \mathbf{P}(F|A) + \mathbf{P}(U) \mathbf{P}(F|U)} \\ &= \frac{(0.3)(0.1)}{(0.3)(0.1) + (0.6)(0.2) + (0.1)(1.0)} = \frac{0.03}{0.03 + 0.12 + 0.10} = \frac{0.03}{0.25} = 0.12. \end{aligned}$$

(3.) Suppose that there are $n \geq 2$ lightbulbs in the theater sign at Broadway and State. Let A_i be the event that the i th lightbulb still functions after one month of use. Assume that the events A_i are independent and have the same probability $\mathbf{P}(A_i) = p$ for all i and for some $0 < p < 1$. Let B be the event that at least two bulbs are still functioning after one month. Express B in terms of the A_i 's Find the probability $\mathbf{P}(B)$. Explain your formula.

There are two main ways to write the event B , “at least two still function.” The event that both i and j bulbs function is $A_i \cap A_j$. The union over all pairs gives the event that “both function for some pair of bulbs”

$$B = \bigcup_{i < j} (A_i \cap A_j).$$

This expression is hard for computing the probability. Easier is the equivalent “not zero remain functional and not exactly one remains functional.” Let Z denote “none remain functional”

$$Z = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.$$

The event E_i that “exactly the i th bulb remains functional” which is “ A_i holds and A_j does not for all $j \neq i$,” or,

$$E_i = A_1^c \cap A_2^c \cap \cdots \cap A_{i-1}^c \cap A_i \cap A_{i+1}^c \cap \cdots \cap A_n^c.$$

Thus, “not zero and not exactly one remain functional” is

$$B = Z^c \cap \left(\bigcup_{i=1}^n E_i \right)^c = \left(Z \cup \bigcup_{i=1}^n E_i \right)^c.$$

By independence $\mathbf{P}(Z) = (1-p)^n$ and $\mathbf{P}(E_i) = p(1-p)^n$ for each i . Since the events Z and E_j are all pairwise disjoint,

$$\mathbf{P}(B) = 1 - \mathbf{P}(Z) - \sum_{i=1}^n \mathbf{P}(E_i) = 1 - (1-p)^n - np(1-p)^{n-1}.$$

(4.) *Three fair dice are rolled. Let A be the event that all three numbers shown on the die are different. Let B be the event that the sum of the three numbers shown is 6. Are A and B disjoint? Are A and B independent? Why?*

Let $\Omega = \{(x_1, x_2, x_3) \in \mathbf{N}^3 : x_i \leq 6 \text{ for each } i\}$ be all outcomes from three rolls, so $|\Omega| = 6^3 = 216$. Then A is the set of permutations of six numbers taken three at a time. Thus $|A| = 6 \cdot 5 \cdot 4$ (six choices for the first roll, five for the second and 4 for the third). B is the event that the rolls add to six, which we enumerate:

$$B = \{(4, 1, 1), (1, 4, 1), (1, 1, 4), (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (2, 2, 2)\}$$

Thus $|B| = 10$. Finally the event that both the sum is six and there are different numbers on each die is

$$A \cap B = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$$

so $|A \cap B| = 6$. Since the intersection is nonempty $A \cap B \neq \emptyset$, the events A and B are NOT DISJOINT. Since

$$\mathbf{P}(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{6}{6^3} \neq \mathbf{P}(A) \mathbf{P}(B) = \frac{6 \cdot 5 \cdot 4}{6^3} \times \frac{10}{6^3} = \frac{200}{6^5}$$

the events A and B are NOT INDEPENDENT.

(5.) *Suppose that you have 26 Scrabble tiles, each with a different letter of the alphabet. In how many ways can you distribute all the tiles into four urns labeled A, B, C and D (where the order of tiles in an urn is not important)? Why? Suppose that the 26 tiles are indistinguishable. In how many ways can you distribute all the tiles into four urns labeled A, B, C and D now? Why?*

If the tiles can be distinguished, then we may count the number of ways to put the tiles into four urns as follows: it’s the number of ways the first tile can be placed into one of the four urns times the number of ways the second tile can be placed into one of the four urns, times the number for the third tile and so on. All, together, there are $\boxed{4^{26}}$ ways to do this.

