Math 5210 § 2. **First Midterm Exam** Name: Solutions

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1. *Let $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{N}\}$ be the set of functions from the natural numbers to the natural numbers. Determine whether $\mathcal{F}$ is countable or uncountable. Why? Let A and B be sets. Define :* $\operatorname{card} A \le \operatorname{card} B$. *Let A and B be sets and $f : A \to B$ be onto. Show* $\operatorname{card} A \ge \operatorname{card} B$.

   $\mathcal{F}$ is uncountable by a Cantor type diagonal argument. Argue by contradiction. If $\mathcal{F}$ were countable, we could enumerate $\mathcal{F} = \{f_1, f_2, f_3, \ldots\}$. However the function $g(n) = f_n(n) + 1$ maps $\mathbb{N}$ to itself but is not in the list because g differs from each f_n since $g(n) \ne f_n(n)$ for $n \in \mathbb{N}$. This is a contradiction since the enumeration failed to list all members of $\mathcal{F}$. Thus $\mathcal{F}$ could not have been countable.

   We say $\operatorname{card} A \le \operatorname{card} B$ if there is a one-to-one function $h : A \to B$.

   If $f : A \to B$ is onto then for each $b \in B$ the preimage set $f^{-1}(\{b\})$ is nonempty. By the axiom of choice, we may choose an element $h(b) \in f^{-1}(\{b\})$. Now the function $h : B \to A$ is one-to-one, so by the definition, $\operatorname{card} B \le \operatorname{card} A$.

2. *Let $\Gamma = \{0.a_1a_2a_3\ldots(\text{base } 10) : a_i \in \{0, 9\}\}$ be the set of decimal fractions whose digits a_n are only 0's or 9's. Write Γ as the countable intersection of closed sets of intervals. Determine whether Γ has measure zero. Determine whether Γ is countable or uncountable. Is the given $f : \Gamma \to \mathbf{R}$ onto? Is f nondecreasing? Is f continuous?*

$$f\big(0.a_1a_2a_3\ldots(\text{base } 10)\big) = \left(0.\frac{a_1}{9}\frac{a_2}{9}\frac{a_3}{9}\ldots(\text{base } 2)\right).$$

   Γ is like the Cantor set Δ, except it is defined by decimals and not ternary fractions. If Γ_n denotes the points where the first n digits are 0's or 9's, then $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$. Note that

$$0.0999\ldots(\text{base } 10) = 0.1000\ldots(\text{base } 10)$$

   If the first digit a_1 of x is anything other than 0 or 9 then

$$0.0999\ldots(\text{base } 10) = 0.1 \le x = 0.a_1a_2a_3\ldots(\text{base } 10) \le 0.8999\ldots(\text{base } 10) = .9(\text{base } 10)$$

   In other words except for $x = .1$ and $x = .9$, x is in the interval $(.1, .9)$ which is not in Γ. Similarly if y is not .01, .09, .91 nor .99, and the first digit is 0 or 9 and the second digit is not 0 nor 9 then

$$y \in (.01, .19) \cup (.90, .99).$$

   which is also not in Γ. Thus we have Γ is the intersection of the closed sets Γ_n consisting of 2^n intervals, each of length 10^{-n}. Γ_{n+1} is obtained from Γ_n by removing the middle open interval which is 0.8 of length of the intervals of Γ_n.

$$\Gamma_1 = [0, .1] \cup [.9, 1]$$

$$\Gamma_2 = [0, .01] \cup [.09, .1] \cup [.9, .91] \cup [.99, 1]$$

$$\Gamma_3 = [0, .001] \cup [.009, .01] \cup [.09, .091] \cup [.099, .1] \cup [0.9, 0.901] \cup [.909, 0.91] \cup [.99, .991] \cup [.999, 1]$$

   To see that Γ has measure zero, it suffices to find a countable collection of open intervals that contains Γ and has total length less than any $\epsilon > 0$. Note that Γ_n is covered by a U_n consisting of 2^n open intervals of length $2 \cdot 10^{-n}$ which have total length

$$2^n \cdot 2 \cdot 10^{-n} = 2 \cdot 5^{-n}.$$

This is less that ϵ by taking n large. For such n, $\Gamma \subset \Gamma_n \subset U_n$ whose total length is less that ϵ. Thus Γ has measure zero.

Γ is in one-to-one correspondence with the set $\mathcal{S}$ of sequences of ones and zeros, thus is uncountable. If $s = (s_1, s_2, s_3, \ldots)$ is such a sequence, then the correspondence $h : \mathcal{S} \to \Gamma$ is given by

$$h(s) = 0.(9s_1)(9s_2)(9s_3)\ldots \text{(base 10)}.$$

The function f is similar to the Cantor function. f is nondecrasing because it preserves order from decimal to the binary expansions. It is onto because each $y \in [0,1]$ has a binary expansion $y = 0.s_1s_2s_3\ldots$ (base 2) which is the image of f, namely $y = f\big(h((s_1, s_2, s_3, \ldots))\big)$. This also shows that if y does not have a unique binary representation, one ending in $y = ***1000\ldots$ and the other ending in $y = ***0111\ldots$ then it is the image of two numbers $y = f(***9000\ldots) = f(***0999\ldots) = f(***1000\ldots)$, which are the endpoints of an excluded open interval of Γ. Extending f to all of $[0,1]$ by making it constant on excluded intervals yields a nondecreasing and onto function $f : [0,1] \to [0,1]$, which must be continuous since it makes no jumps as it is onto. The restriction of f to Γ yields a continuous function on Γ,

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *The function $d(x,y) = |x^2 - y^2|$ is a metric on the real numbers.*

   FALSE. The positive definite condition for metrics is $d(x,y) = 0$ if and only if $x = y$. But here, $d(3,-3) = |3^2 - (-3)^2| = 0$ so the positive definite condition for metrics fails.

   (b) [7] STATEMENT: *Let ℓ_∞ denote the space of bounded real sequences. Then $\|(x_1, x_2, x_3, \ldots)\| = \sup\limits_{i\in\mathbb{N}} |x_i|$ is a norm on ℓ_∞.*

   TRUE. The three conditions for a norm hold: For any $x, y \in \ell_\infty$,

   i. $\|x\| = \sup_{i\in\mathbb{N}} |x_i| \in [0, \infty)$;
   ii. For $a \in \mathbf{R}$, $\|ax\| = \sup_{i\in\mathbb{N}} |ax_i| = \sup_{i\in\mathbb{N}} |a||x_i| = |a| \sup_{i\in\mathbb{N}} |x_i| = |a|\,\|x\|$;
   iii. $\|x+y\| = \sup_{i\in\mathbb{N}} |x_i+y_i| \le \sup_{i\in\mathbb{N}}(|x_i|+|y_i|) \le \sup_{i\in\mathbb{N}} |x_i| + \sup_{i\in\mathbb{N}} |y_i| = \|x\|+\|y\|$.

   (c) STATEMENT: *Let $\mathbb{V}$ be a real vector space with inner product $\langle \bullet, \bullet \rangle$, $\xi \in \mathbb{V}$ a nonzero vector and $r > 0$. Then there is only one vector that maximizes the function $f(y) = \langle y, \xi \rangle$ among vectors that satisfy $\|y\| \le r$.*

   TRUE. By the Cauchy Schwarz inequality, for any $y \in \mathbb{V}$ such that $\|y\| \le r$ we have

   $$f(y) = \langle y, \xi \rangle \le |\langle y, \xi \rangle| \le \|y\|\,\|\xi\| \le r\|\xi\|.$$

   If there were such y where $f(y)$ takes its limiting value, we would have equality in the Schwarz inequality which implies $y = k\xi$ for some $k \in \mathbf{R}$. In fact $f(k\xi) = k\langle \xi, \xi \rangle = k\|\xi\|^2 = r\|\xi\|$ takes its limiting value if $k = r/\|\xi\|$. Thus this $y = k\xi$ is the unique maximizer.

4. *Let $\mathcal{C} = \{(a_i) : a_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N} \text{ and } (a_i) \text{ is a Cauchy Sequence}\}$ be the set of Cauchy sequences of rationals and $\mathcal{N} = \{(a_i) : a_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N}, a_i \to 0 \text{ as } i \to \infty\}$ the set of null sequences of rationals. Consider the quotient space $\mathcal{R} = \mathcal{C}/\mathcal{N}$ where the Cauchy sequences (a_i) and (b_i) are equivalent if $(a_i - b_i) \in \mathcal{N}$. Give the definition of multiplication "$\times$" on $\mathcal{R}$. Show that $x, y \in \mathcal{R}$ implies that $x \times y \in \mathcal{R}$. Check that $x \times y$ is well defined.*

Let $x = [(a_i)]$ and $y = [(b_i)]$ be two equivalence classes in $\mathcal{R}$. Multiplication is defined componentwise

$$x \times y = [(a_i)] \times [(b_i)] = [(a_i b_i)].$$

(a_ib_i) is a Cauchy sequence so that $x \times y \in \mathcal{R}$. To see it, recall that Cauchy sequences are bounded: there are rational $M_1, M_2 < \infty$ such that

$$|a_i| \le M_1 \quad \text{and} \quad |b_i| \le M_2 \qquad \text{for all } i \in \mathbb{N}.$$

Choose rational $\varepsilon > 0$. Since (a_i) and (b_i) are Cauchy sequences, there are $N_1, N_2 \in \mathbb{N}$ such that

$$|a_i - a_j| \le \frac{\varepsilon}{2M_2 + 1} \text{ whenever } i, j \ge N_1 \quad \text{and} \quad |b_i - b_j| \le \frac{\varepsilon}{2M_1 + 1} \text{ whenever } i, j \ge N_1.$$

Let $N_3 = \max\{N_1, N_2\}$. For any $i, j \ge N_3$ there holds

$$\begin{aligned} |a_ib_i - a_jb_j| &= |a_ib_i - a_ib_j + a_ib_j - a_jb_j| \\ &= |a_i(b_i - b_j) + (a_i - a_j)b_j| \\ &\le |a_i|\,|b_i - b_j| + |b_j|\,|a_i - a_j| \\ &\le M_1 \frac{\varepsilon}{2M_1 + 1} + M_2 \frac{\varepsilon}{2M_2 + 1} < \varepsilon. \end{aligned}$$

Thus (a_ib_i) is a Cauchy sequence of rationals as claimed.

Choose equivalent Cauchy sequences $(a'_i) \sim (a_i)$ and $(b'_i) \sim (b_i)$ so that $a'_i - a_i \to 0$ and $b'_i - b_i \to 0$ as $i \to \infty$. To show that multiplication is well defined we claim $(a'_ib'_i) \sim (a_ib_i)$. Using the fact if (p_i) is bounded and $q_i \to 0$ as $i \to \infty$ implies $p_iq_i \to 0$ as $i \to \infty$, we have from the boundedness of Cauchy sequences, as $i \to \infty$,

$$a'_ib'_i - a_ib_i = a'_ib'_i - a'_ib_i + a'_ib_i - a_ib_i = a'_i(b'_i - b_i) + b_i(a'_i - a_i) \to 0 + 0.$$

Thus multiplication doesn't depend on representatives so is well defined.

5. *Let $\mathcal{R} = \mathcal{C}/\mathcal{N}$ with "+" and "×" as in Problem 4 and let $[(a_i)], [(b_i)], [(c_i)] \in \mathcal{R}$. What is the definition of $[(a_i)] > [(b_i)]$? Suppose that $[(a_i)] > [(b_i)]$ and $[(c_i)] > 0$. Show using your definition that $[(a_i)] \times [(c_i)] > [(b_i)] \times [(c_i)]$.*

The definition of ordering $[(a_i)] > [(b_i)]$ is that there is a rational $\varepsilon_1 > 0$ and $N_1 \in \mathbb{N}$ such that

$$a_i - b_i > \varepsilon_1 \qquad \text{whenever } i \ge N_1.$$

Similarly, $[(c_i)] > 0^* = [(\bar{0})]$ means there is a rational $\varepsilon_2 > 0$ and $N_2 \in \mathbb{N}$ such that

$$c_i > \varepsilon_2 \qquad \text{whenever } i \ge N_2.$$

Let $N_3 = \max\{N_1, N_2\}$. Then

$$(a_ic_i) - (b_ic_i) = (a_i - b_i)c_i > \varepsilon_1\varepsilon_2 \qquad \text{whenever } i \ge N_3.$$

Thus by definition, $[(a_i)] \times [(c_i)] = [(a_ic_i)] > [(b_ic_i)] = [(b_i)] \times [(c_i)]$.