# Real Analysis 5210

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**1.** Let $X$ and $Y$ be metric spaces, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that the set $\{x \in X : f(x) = g(x)\}$ is dense in $X$.

Show that $f(x) = g(x)$ for each $x \in X$.

**2.** Let $X$, $Y$ be metric spaces. Assume also that $X$ is compact. Let $f : X \to Y$ be a continuous map.

Prove that if $D$ is a closed subset of $X$, then $f(D)$ is a closed subset of $Y$.

**3.** Let $f_n : [0, 1] \to \mathbb{R}$ be defined by:

$$f_n(x) = \frac{x^n}{1 + x^2}$$

.

Show that $f_n$ converges pointwise on $[0, 1]$, but this convergence is not uniform.

**4.** Show that

$$\lim_{n \to \infty} \int_0^\infty \sin(nx) e^{(-nx^2)}\, dx = 0$$

**5.** Let $A = \{u_n\}_{n=1}^\infty$ be an orthonormal set of vectors in a Hilbert space $H$. Show that $A$ is not a compact subset of $H$.

(Hint: compute the distance between any two vectors in $A$).

**6.** Let $g : [0, 1] \to \mathbb{R}$ be a bounded measurable function. Define $T : L^1([0, 1]) \to \mathbb{R}$ by $F(f) = \int_0^1 f(x)g(x)\, dx$.

Show that $T$ is a bounded linear functional on $L^1([0, 1])$.

**7.** Let $u_n : \mathbb{Z} \to \mathbb{R}$ be defined by

$$u_n(k) = \begin{cases} 1, & \text{if } k \geq 0 \text{ and } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Show that $A = \{u_n\}_{n=1}^\infty$ is an orthonormal system in $L^2(\mathbb{Z})$, but it is not a Hilbert basis of $L^2(\mathbb{Z})$.

**8.** Let $X \subset C([-1, 1])$ be the subspace consisting of polynomials. Consider the operator $T : X \to X$, $T(f) = f'$ (the derivative of $f$). Show $T$ is linear and onto $X$. Is $T$ bounded? Justify your answer.