Math 5210 Final Exam - 4/30/13 Name

Each sub-problem worth 10 points

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1. (a) Let (X, d) be a metric space. Define what it means for it to be *complete*.

   (b) Recall that an infinite series $\sum_{n=1}^{\infty} a_n$ of real numbers is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges. Assume that we know that $\mathbb{R}$ is a complete metric space. Prove that every abolutely convergent series $\sum_{1}^{\infty} a_n$ of real numbers is convergent.

2. (a) State a version Stone-Weierstrass Theorem that implies that the trigonometric polynomials

$$P(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

   are dense in the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ which are periodic with period 2π. You don't need to check the details of the implication.

   (b) Prove that if $f : \mathbb{R} \to \mathbb{C}$ is continuous, periodic of period 2π, and

$$\int_{-\pi}^{\pi} f(x)e^{inx}dx = 0 \text{ for all } n \in \mathbb{Z},$$

   then $f = 0$.
   Suggestion: Prove that $\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} f(x)\overline{f(x)}dx = 0$ by approximating f by trigonometric polynomials $P(x)$ and using the corresponding approximation $\int_{-\pi}^{\pi} P(x)\overline{f(x)}dx$ of the integral.

3. (a) Let $A \subset \mathbb{R}$. Define the *outer measure* $m^*(A)$ of A, and prove that whenever $A \subset B$, $m^*(A) \leq m^*(B)$.

(b) Let $E \subset \mathbb{R}$. Define what it means for E to be *measurable.*

(c) Prove that a set of outer measure zero is always measurable. You may assume that outer measure is sub-additive.

4. (a) Define what is meant by a *simple function* and by the *integral* of a simple function.

(b) Define $\int_E f$ for f a bounded measurable function on a measurable set E of finite measure. Then define $\int_E f$ for f a non-negative measurable function on an arbitrary measurable set E.

(c) Let f_n be a sequence of non-negative measurable functions on a measurable set E, suppose $f_n(x) \to f(x)$ a.e on E. What is the relation between

$$\int_E f \quad \text{and} \quad \underline{\lim} \int_E f_n \ ?$$

Give an example that shows that the inequality can be strict.