Math 5210 Final Exam - 4/30/15 Name

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1. Let (X, d) be a metric space.

   (a) (10 pts) Define what it means for X to be *complete*.

   (b) (10 pts) Suppose (X, d) is complete and suppose that $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \leq \frac{1}{2^n}$. Prove that $\{x_n\}$ converges.

(c) (10 pts) Suppose (X, d) is complete and $f : X \to X$ satisfies

$$d(f(x), f(y)) \leq \frac{d(x, y)}{2}.$$

Prove that f has a fixed point.

(d) (10 pts) Prove that this fixed point is unique.

2. (a) (10 pts) Let (X, d) be a metric space and let $\mathcal{F}$ be a collection of real-valued functions on X. Define what it means for $\mathcal{F}$ to be *equicontinuous.*

(b) (10 pts) Let $f_n : [0, 2\pi] \to \mathbb{R}$ be defined by $f_n(x) = \int_0^x \sin(n^2 t^3) dt$. Prove that the collection $\{f_n\}$ is equicontinuous.

3. (a) (5 points) Let M be a set and let $\omega : 2^M \to [0, \infty]$. Define what it means for ω to be an (abstract) *outer measure.*

(b) (5 pts) Suppose $\omega : 2^M \to [0, \infty]$ is an (abstract) outer measure. Define what it means for a subset $E \subset M$ to be *measurable* (with respect ot ω).

(c) (10 pts) Let $A \subset \mathbb{R}$. Define $m^*(A)$, the *Lebesgue outer measure* of A.

(d) (10 pts) Suppose $A \subset \mathbb{R}$ is countable. Prove that $m^*(A) = 0$.

(e) (10 pts) Prove that if $E \subset \mathbb{R}$ and $m^*(E) = 0$, then E is Lebesgue measurable.