Math 5210 Midterm 2 - 4/3/13 Name

Each sub-problem worth 10 points

1. (a) Let (X, d) be a metric space and let $\mathcal{F}$ be a family of continuous real valued functions on X. Define what it means for $\mathcal{F}$ to be *equicontinuous*.

   (b) Suppose $f_n : [0,1] \to \mathbb{R}$ is a uniformly bounded sequence of continuous functions: there exists a constant C so that $|f_n(x)| \leq C$ for all $x \in [0,1]$ and for all n. Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^x f_n(t)dt.$$

   Prove that the family $\{g_n\}$ is equicontinuous on $[0,1]$.

2. (a) State the Stone-Weierstrass Theorem for real-valued functions on a compact metric space X.

(b) Let $\mathcal{A}$ be the algebra of polynomials of even degree in $[-1, 1]$. The uniform closure of this algebra consists of even functions, so it is not the whole space of continuous functions on $[-1, 1]$. Which hypothesis of the Stone-Weierstrass theorem is not satisfied by $\mathcal{A}$?

3. (a) Let $U \subset \mathbb{R}^m$ be an open set, let $f : U \to \mathbb{R}^n$, let $x \in U$. Define what it means for f to be *differentiable* at x. Then define what it means for f to be *continuously differentiable* in U.

(b) State the *Inverse Funcion Theorem.*

(c) Ler $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^2 - y^2, 2xy)$. Find all points (x, y) in the domain where the Inverse Function Theorem guarantees the existence of a local inverse function.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable and periodic of period 2π: $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.

   (a) Define what is meant by the *Fourier Series* of f. Include in your definition a formula for the *Fourier Coefficents* c_n of f.

   (b) State two parts of *Parseval's Theorem*: in what sense do the partial sums $s_N(f, x)$ of the Fourier series converge to f? How is the sum $\Sigma_{-\infty}^{\infty}|c_n|^2$ computed in terms of a single integral involving f?

   (c) Let $f(x) = x$ for $-\pi \leq x < \pi$, extended to $\mathbb{R}$ to be periodic of period 2π, and suppose that you know that its Fourier series is

   $$-i \sum_{n \neq 0} \frac{e^{inx}}{n}.$$

   Find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.