Math 5210 § 1. **Final Exam** Name:______________
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This exam is closed book and closed note. Pick 6 of 10 of the problems to hand in. Each problem is worth a total of 25 points. Good Luck!

1. (a) [10] If X is compact and $f : X \to Y$ is continuous, prove $f(X)$ is compact.
   (b) [10] If X is connected and $f : X \to Y$ is continuous, prove that $f(X)$ is connected.
   (c) [5] If X is compact and connected and $f : X \to Y$ is continuous, what type of subset must $f(X)$ be? Explain! (Be precise!)

2. [25] Let X be compact and Y be complete. Consider the function metric space

$$\mathcal{C}(X, Y) = \{f : X \to Y : f \text{ is continuous}\}, \qquad d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x))$$

   We've shown that $\mathcal{C}(X, Y)$ is a metric space. Show that $\mathcal{C}(X, Y)$ is complete.

3. Consider $\mathcal{C}([0, 1]) = \{f : [0, 1] \to \mathbb{R} : f \text{ is continuous}\}$ with $\| \bullet \|_{\text{sup}}$.
   (a) [10] Exhibit a closed and bounded subset of $\mathcal{C}([0, 1])$ which is not compact (and prove it's not compact).
   (b) [10] Carefully state the Arzela-Ascoli Theorem which characterizes the compact subsets of $\mathcal{C}([0, 1])$. (Proof not required.)
   (c) [5] If for L and M fixed, $f_k \in \mathcal{C}([0, 1])$ such that

$$\|f_k\|_{\text{sup}} \le M, \qquad |f_k(x) - f_k(y)| \le L|x - y| \qquad \text{for all } x, y \in [0, 1] \text{ and for all } k,$$

   does $\{f_n\}$ have a convergent subsequence? Explain.

4. [25] Let $B_1(0) \subset \mathbb{R}^n$ and $f : B_1(0) \to \mathbb{R}^n$ be differentiable on its domain such that for some fixed M, the derivative operator $d_x f$ satisfies $\|d_x f\|_{\text{op}} \le M$ for all $x \in B_1(0)$. Prove that f is Lipschitz continuous with estimate

$$|f(x) - f(y)| \le M|x - y| \qquad \text{for all } x, y \in B_1(0).$$

   Hint: Integrate $g'(t)$ for $g(t) = f((1 - t)y + tx)$. Use the FTC, chain rule and estimate!

5. [25] State and prove the Contraction Mapping Theorem.

6. Let (X, d) be a metric space.
   (a) [5] What is an *outer measure* μ? (*i.e.,* give the definition.)
   (b) [5] What makes an outer measure a *metric outer measure*?
   (c) [5] Let $E \subset X$, μ an outer measure. What does it mean for E to be *measurable* (in Strichartz' "splitting cond.")?
   (d) [5] Let E_1, E_2 be measurable disjoint subsets of X. Prove $E_1 \cup E_3$ is measurable, using def. (c).
   (e) [5] State the theorem we proved that guarantees there are lots of measurable sets for any metric outer measure.

7. (a) [10] Give an example of a sequence of functions $f_n : [0,1] \to \mathbb{R}$ such that each f_n is continuous, $\{f_n(x)\}$ converges for each x, but to a function $f(x)$ which is not continuous.

   (b) [15] If $f_n : [0,1] \to \mathbb{R}$ such that each f_n is measurable (say for Lebesgue measure m), and if $f_n(x) \to f(x)$ for each $x \in [0,1]$, prove $f(x)$ is also measurable.

8. Our favorite estimate in the class (after the Δ inequality) is $\left|\int f\right| \le \int |f|$.

   (a) [13] Let $f : [a,b] \to X_{\text{Banach}}$ be continuous. Using the definition of the Riemann Integral, prove

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

   (b) [12] Let $(X, \mathcal{F}, \mu)$ be a measure space and $f \in \mathcal{L}^1(X, \mu)$. Prove (using f^+, f^-) that

$$\left|\int_X f\,d\mu\right| \le \int_X |f|\,d\mu.$$

9. (a) [10] Let $f_n, f : [0,1] \to \mathbb{R}$, f_n be continuous and $f_n \to f$ uniformly. Use the (8a) theorem to prove

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

   (b) [10] Let $f_n, f : [0,1] \to \mathbb{R}$, f_n measurable (with respect to Lebesgue measure m), and $f_n \to f$ *a.e.* State the Lebesgue Dominated Convergence Theorem, which gives a stronger version of (9a).

   (c) [5] Exhibit a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that $f_n(x) \to f(x)$ pointwise on $[0,1]$, but not uniformly, but such that the integrals do <u>converge</u> to $\int_0^1 f(x)\,dx$ and LDCT <u>does</u> hold, even though (9a) fails.

10. We used polar coordinates to prove that $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$. Compute (and justify!) the following two limits:

   (a) [12]

$$\lim_{t\to 0+} \int_{-\infty}^{\infty} \frac{e^{-x^2 - t^2 x^4}}{\sqrt{1+t^2}}\,dx.$$

   (b) [13]

$$\lim_{t\to 0+} \int_{-\infty}^{\infty} t e^{-t^2 x^2}\,dx \qquad \text{(Careful!!)}$$